Functional Analysis I

1st week

Linear spaces, subspaces, basis and dimension of a linear space.

2nd week

Normed spaces, open and closed ball of a normed space. Open and closed sets of a normed space. Continuity of the norm and of addition/scalar multiplication. Convergent and Cauchy sequences. The definition of a Banach space. Examples of Banach spaces: $\mathbb{R}^n$, $C([a,b)]$, l^p for $1 \leq p < \infty$.

3rd/4th week

Finite dimensional normed spaces. Equivalent norms. We proved that all norms on a finite dimensional space are equivalent and then that every finite dimensional space is a Banach space. We then proved that if for a subspace Y of a normed space X we have that $Y^\circ \neq \emptyset$, then $Y = X$. We concluded with the very important Theorem that the dimension of a Banach space is either finite or uncountable. Bounded linear operators.

5th week

Main properties of bounded linear operators. Examples: the Voltera integral operator and the right shift. The norm of a bounded linear operator.

6th week

We gave the definition of codimension of a subspace and we proved that the codimension of the kernel of a linear functional is 1. The separability of finite dimensional spaces, of c_{00} and of l^p , for $1 \leq p < \infty$. The non-separability of l^∞.

7th week

We stated the Hahn-Banach Theorem and proved some of its corollaries.

8th week

Using the corollaries of the Hahn-Banach theorem we proved that if X^* is separable, then X is separable and that X is finite dimensional if and only if its closed unit ball is compact. We begun the proof of the Hahn-Banach theorem.

9th week

We finished the proof of the Hahn-Banach theorm. We showed the X may be embedded to its second dual. We also showed that if $p > 1$, then $(l^p)^* = l^q$ for all $\frac{1}{p} + \frac{1}{q} = 1$ and hence that if $p > 1$, then l^p is reflexive.

10th week

We proved the Uniform Boundedness theorem, the Open Mapping Theorem and the Closed Graph Theorem.

11th week

We gave the definition of a Hilbert space H, proved the Cauchy-Schwarz inequality and showed that for every convex closed subset of H and x_0 there exists a unique point $y \in H$ such that $\text{dist}(x_0, Y) = \|x_0 - y\|$. We defined the notion of the orthogonal complement of a subspace of a Hilbert space and showed that H may be decomposed as the direct sum of a closed subspace and its orthogonal complement.

12th week

We gave the definition of an orthonormal set, discussed some of its properties and proved Besel's ineaquality. We gave the definition of a complete orthonormal set (orthonormal basis). We proved a characterization for complete orthonormal sets. We described the Gram-Schmidt method.