Functional Analysis I
(Solutions of problem sheet 5)

Exercise 1. *Let X be a normed space. Show that for any $x \in X$ there exists $f \in X^*$, such that $f(x) = \|x\|^2$ and $\|f\| = \|x\|$.*

Solution. Let $x \in X$. Set $M = \langle x \rangle$ and define $f : M \to \mathbb{R}$ by

$$f(\lambda x) = \lambda \|x\|^2 \,, \text{ for all } \lambda \in \mathbb{R} \,.$$

Then f is a bounded linear functional on M with $\|f\| = \|x\|$. By the Hahn-Banach theorem f may be extended to bounded linear functional on X. Denoting this extenesion again by f we have the desired conclusion. □

Exercise 2. *Let X be a normed space and Y be a proper closed linear subspace of X. If $x_0 \notin Y$ show that there exists $f \in X^*$, such that*

$$\|f\| = \frac{1}{d(x_0, Y)} \,, f(x_0) = 1 \text{ and } f(y) = 0, \text{ for all } y \in Y \,.$$

Solution. Let $M = \langle Y \cup \{x_0\} \rangle$ and $f : M \to \mathbb{R}$ defined by

$$f(y + \lambda x_0) = \lambda \,, \text{ for all } y \in Y, \lambda \in \mathbb{R} \,.$$

Then f is obviously linear. Moreover

$$|f(y + \lambda x_0)| = |\lambda| \leq \frac{1}{d(x_0, Y)} \|y + \lambda x_0\| \,, \text{ for all } y \in Y, \lambda \in \mathbb{R}$$

and thus $\|f\| \leq \frac{1}{d(x_0,Y)}$.

We will show that $\|f\| \geq \frac{1}{d(x_0,Y)}$. To this end let (y_n) be a sequence in Y such that

$$\lim \|y_n - x_0\|| = d(x_0, Y) \,.$$

Then

$$\|f\| = \sup_{0 \neq z \in M} \frac{|f(z)|}{\|z\|} \geq \frac{|f(y_n - x_0)|}{\|y_n - x_0\|} \to \frac{1}{d(x_0, Y)}$$

and hence $\|f\| \geq \frac{1}{d(x_0,Y)}$. Combining the above we have that

$$\|f\| = \frac{1}{d(x_0, Y)} \,, f(x_0) = 1 \text{ and } f(y) = 0, \text{ for all } y \in Y \,.$$

By the Hahn-Banach theorem f may be extended to bounded linear functional on X. Denoting this extenesion again by f we have the desired conclusion. □

Exercise 3. *Let M be a subset of the nornmed space X. Show that $x_0 \in X$ belongs to the set $\overline{\langle M \rangle}$ if and only if $f(x_0) = 0$, for all $f \in X^*$, such that $f|_M = 0$.*

Solution. If $x_0 \in \overline{\langle M \rangle}$ and $f \in X^*$, is such that $f|_M = 0$, then by the continuity of f we have that $f(x_0) = 0$.

Conversly if $x_0 \notin \overline{\langle M \rangle}$ by the Hahn-Banach theorem there exists $f \in X^*$, such that $f(x_0) \neq 0$ and $f(z) = 0$, for all $z \in \overline{\langle M \rangle}$ and hence $f|_M = 0$. Therefore if $f(x_0) = 0$, for all $f \in X^*$, such that $f|_M = 0$, then $x_0 \in \overline{\langle M \rangle}$. □

Exercise 4. *Let X be a normed space and Y be a closed linear subspace of X. We say that $z \in X$ is orthogonal to Y, and write $z \perp Y$, if $\operatorname{dist}(z, Y) = \|z\|$. Show that $z \perp Y$ if and only if there exists $0 \neq f \in X^*$ such that $f|_Y = 0$ and $|f(z)| = \|f\|\|z\|$. Moreover show that if $Y \neq X$ and $x_0 \notin Y$, then $y_0 \in Y$ is the nearest point of Y to x (i.e. $0 < \operatorname{dist}(x_0, Y) = \|x_0 - y_0\|$) if and only if $x_0 - y_0 \perp Y$.*

Solution. Assume that $\operatorname{dist}(z, Y) = \|z\|$. Since $z \notin Y$, from the second corollary of the Hahn-Banach theorem there exists $f \in X^*$, such that $\|f\| = 1$, $f|_Y = 0$ and $f(z) = \operatorname{dist}(z, Y)$ and hence for ths particular f we have that $|f(z)| = \|f\|\|z\|$.

Conversly if there exists $0 \neq f \in X^*$ such that $f|_Y = 0$ and $|f(z)| = \|f\|\|z\|$, then for all $y \in Y$ we have that $\|z + y\|\|f\| \geq |f(z + y)| = |f(z)|\|f\|\|z\|$. Therefore $\|z\| = \operatorname{dist}(x, Y)$.

It is easy to show that since $y_0 \in Y$, then $\operatorname{dist}(x_0 - y_0, Y) = \operatorname{dist}(x_0, Y)$. Hence $\operatorname{dist}(x_0, Y) = \|x_0 - y_0\|$ if and only if $\operatorname{dist}(x_0 - y_0, Y) = \|x_0 - y_0\|$ i.e. if and only if $x_0 - y_0 \perp Y$. □

Exercise 5. *Let X be a normed space, Y be a linear subspace of X and*

$$F = \{f \in X^* : \|f\| \leq 1 \, , f|_Y = 0\} \, .$$

Prove that:

(i) For any $x \in X$

$$\operatorname{dist}(x, Y) = \sup_{f \in F} |f(x)| \, .$$

(ii)

$$\overline{Y} = \bigcap_{f \in F} \ker f \, .$$

(Hint: use exercise 2.)

Solution. (i) Since for any $f \in F$ and $y \in Y$we have that $|f(x)| = |f(x - y)| \leq \|x - y\|$ taking supremum over all $f \in F$ and then infimum over all $y \in Y$ we have that $\sup_{f \in F} |f(x)| \leq \operatorname{dist}(x, Y)$.

For the reverse inequality, if $\operatorname{dist}(x, Y) > 0$, using f from exercise 2, we have that

$$\operatorname{dist}(x, Y) = \frac{|f(x)|}{\|f\|}$$

and since $\frac{f}{\|f\|} \in F$ we have that $\operatorname{dist}(x, Y) \leq \sup_{f\in F} |f(x)|$.
(ii) If $Y = X$ we have nothing to prove. Otherwise if $f \in F$ then$\overline{Y} \subseteq \overline{\ker f} = \ker f$ and hence $\overline{Y} \subseteq \bigcap_{f\in F} \ker f$.

For the reverse inclusion if $x \notin \overline{Y}$, from the second corollary of the Hahn-Banach theorem there exists $f \in X^*$ such that $\|f\| = 1$, $f|_Y = 0$ and $x \notin \ker f$. Hence $x \notin \bigcap_{f\in F} \ker f$ and thus $\bigcap_{f\in F} \ker f \subseteq \overline{Y}$. □