We require that the strain energy density function of the cartilage mixture (ground substanse + collagen fiber), be positive.

For a loading symmetric about the 1 axis, we have the expression of the strain energy density, in a principal stress - strain system, written as

$$\frac{1}{2}\sigma_{11}\varepsilon_{11} + \frac{1}{2}\sigma_{22}\varepsilon_{22} + \frac{1}{2}\sigma_{33}\varepsilon_{33} =$$

$$\frac{1}{2}E_{11}^{car}\varepsilon_{11}^{2} + 2E_{12}^{car}\varepsilon_{11}\varepsilon_{22} + \left(E_{22}^{car} + E_{23}^{car}\right)\varepsilon_{22}^{2} > 0 \quad (1)$$

which is required to be positive. This leads to the restrictions

$$E_{11}^{car} > 0 \quad , \quad 2\left(E_{12}^{car}\right)^{2} - E_{11}^{car}\left(E_{22}^{car} + E_{23}^{car}\right) < 0 \quad (2)$$

Inequalities (2) indicate that the elastic constants of the newly created material (obtained by a uniaxial extension of the fiber+matrix along the 1 direction), cannot be completely independent.

<u>Nonlinear stress-strain response of collagen fibers</u>

We consider the strain energy density function

$$w_c = \frac{K_c}{2k_c}\left[\exp\left(k_c L^2 - 1\right)\right] \quad (1)$$

where

$$L = \langle \underset{\sim}{\varepsilon} : \underset{\sim}{M_c} \rangle = \langle \varepsilon_{11}\cos^2\underline{\theta} + \varepsilon_{22}\sin^2\underline{\theta} \rangle \quad (2)$$

L: the enlongation of ~~of~~ a fiber, oriented at an angle $\underline{\theta}$ with the horizontal plane

$\langle\rangle$: posite part. If the content of the $\langle\rangle$ is negative, this is zero.

$\underset{\sim}{\varepsilon}$: strain tensor (small strain tensor)

$\underset{\sim}{M}_c$: directional $2^{\underline{nd}}$ order tensor of a single fiber

For the whole bundle of fibers, the new strain energy, denoted by $W^c(\underset{\sim}{\varepsilon})$ is given by the relation

$$W^c(\underset{\sim}{\varepsilon}) = n^c \int_{S^2} w_c(L) \frac{dS}{4\pi} \quad (3)$$

S^2: surface of the unit sphere

dS: differential surface on the finite surface of the unit sphere.

n^c: volume fraction that the collagen fibers occupy in the mixture (matrix + fibers + fluid?)

For the stresses of the bundle of fibers we write

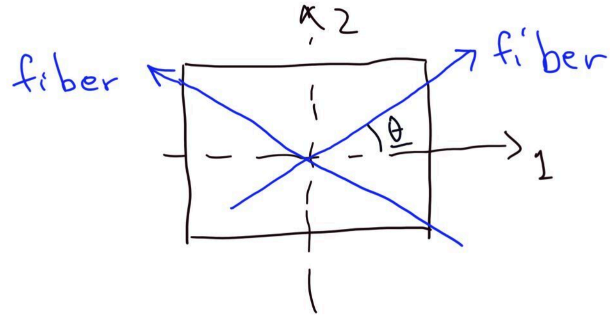
$$\underset{\sim}{\sigma}^c = \frac{\partial W^c(\underset{\sim}{\varepsilon})}{\partial \underset{\sim}{\varepsilon}} \quad (4)$$

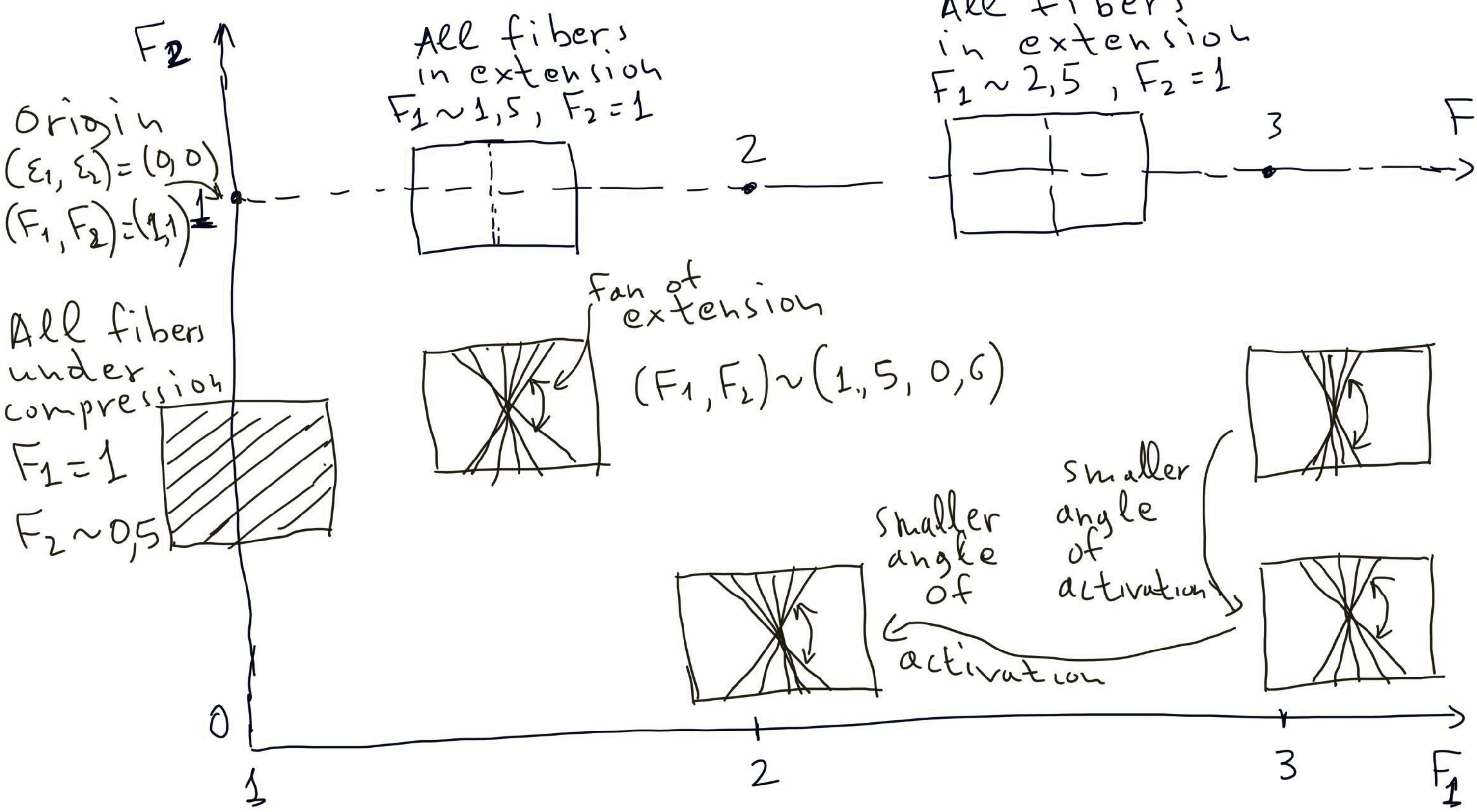
for the non-linear stress-strain relation, that is imposed be the exponential form of the strain energy densit $w_c(\underset{\sim}{\varepsilon})$ of a single collagen fiber.

<u>Fans of fiber activation under a biaxial extension-compression deformation</u>

We are in the plane 1,2. The unit directional vector of a fiber is

$$\underline{\vec{m}} = (\cos\underline{\theta}, \sin\underline{\theta}, 0)$$

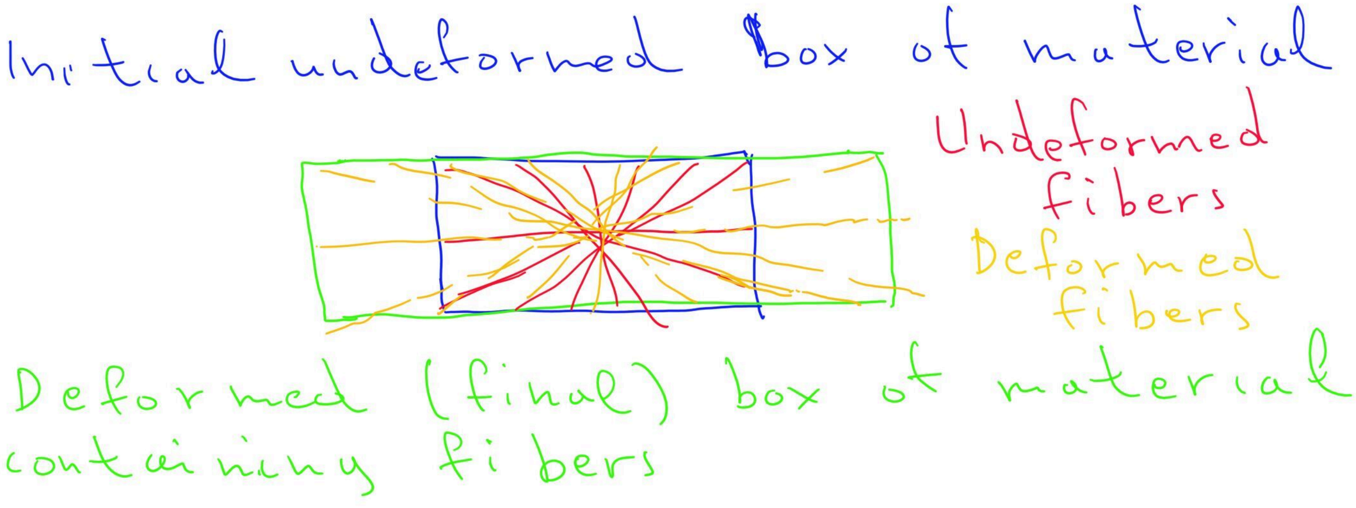




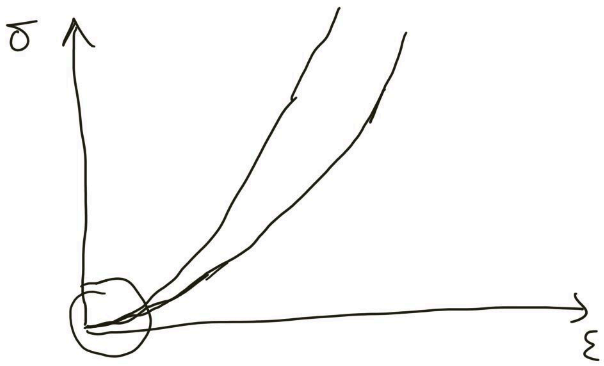
For this particular biaxial deformation, we have

$$F_1 = 1 + \varepsilon_1 \geq 1 \quad (\text{extension}) \quad \varepsilon_1 \geqslant 0$$

$$F_2 = 1 + \varepsilon_2 \leq 1 \quad (\text{compression}) \quad \varepsilon_2 \leq 0$$



White colour within the fan, indicates extension of the fibers.
Dark colour within the fan, indicates compression of the fibers.
Approaching the origin, from the right or from below along the axes, we encounter abrupt changes, from extension to compression, for all the fibers. They are all activated when all are under extension and they are all inactive, when they are all under compression. So we have a discontinuous transfer from inactivation to activation. These discontinuities may be responsible for the toe in the stress-strain diagramme of the soft tissues containing collagen fibers.



We may encounter geometricall non-linear response in the initial stages of deformation

The transformed (deformed) length of an undeformed unit fiber straight line segment $\underline{\vec{m}}\,(\cos\underline{\theta}, \sin\underline{\theta}, 0)$, has square

$$\ell^2 = \underline{\vec{m}} \cdot \underset{\sim}{C} \cdot \underline{\vec{m}} = F_1^2 + \sin^2\underline{\theta}\left(F_2^2 - F_1^2\right)$$

We require that $\ell > 0$ and ~~form~~ from the above relation we find the

the fan $(-\underline{\theta}, \underline{\theta})$ with

$$\sin^2\underline{\theta} = (F_1^2 - 1) / (F_1^2 - F_2^2)$$

contains ~~the fibers~~ the fibers that are under extension.

Close to the origin, i.e. when $F_1 \to 1$, $F_2 \to 1$, the above relation takes the form

$$\sin\underline{\theta} = \frac{1}{\sqrt{1 - \frac{\varepsilon_2}{\varepsilon_1}}}$$

which indicates abrupt changes in the fans of activation (e.g. when $\varepsilon_1 \to 0$ the denominator becomes undefined).