Partition of the tissue (soft tissue) into phases

Thre phases are considered:

1) One solid phase (S), which mainly consists of collagen fibers.
2) One extrafibrillar (EF) fluid phase, consisting of water, ions and proteoglycans
3) One IntraFibrillar (IF) fluid phase, consisting of water and ions

A bath is also considered, surrounding the extrafibrillar fluid phase

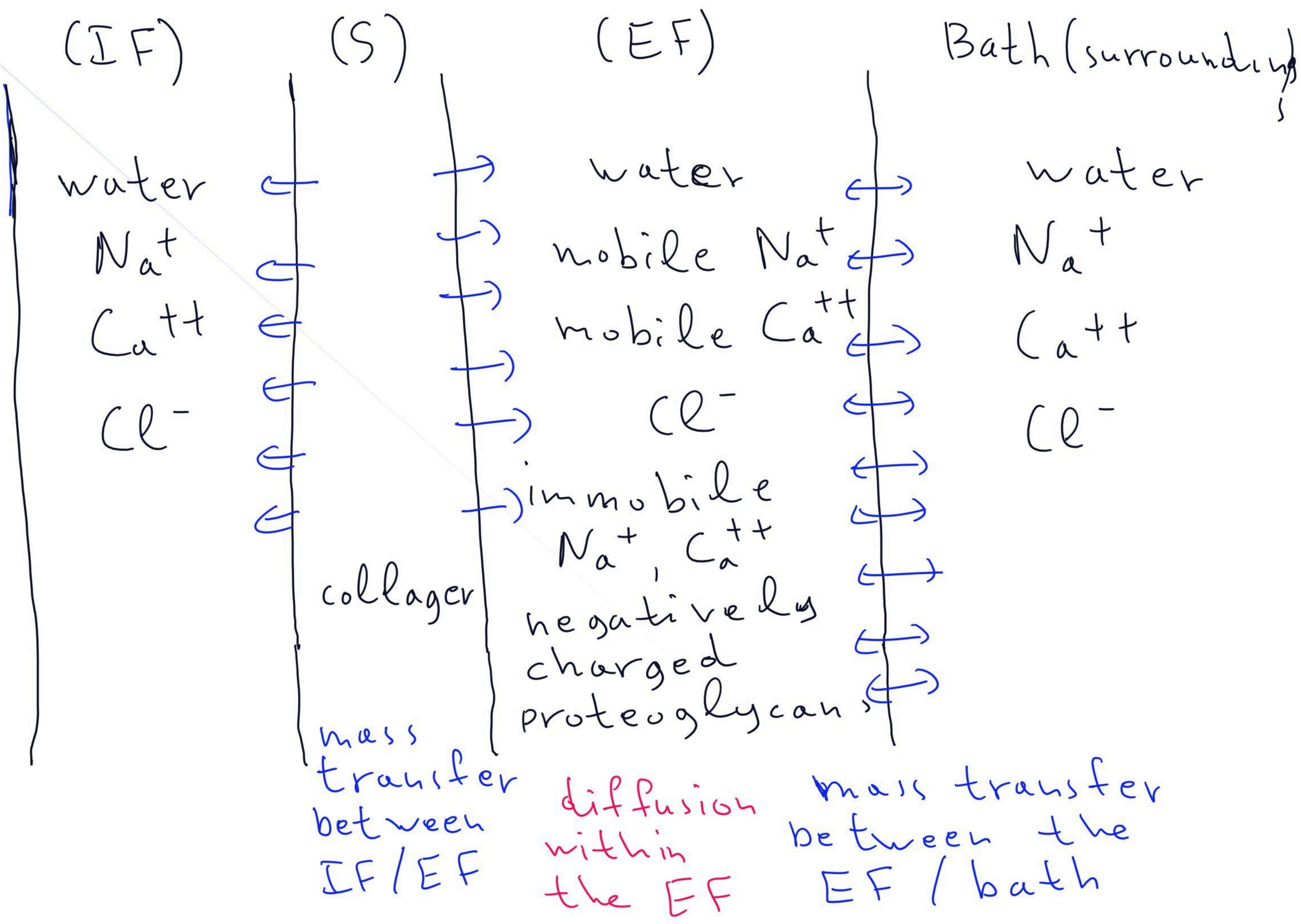
Via a compact notation we can write

$$S = \{c\} \qquad c: \text{collagen}$$

$$I = \{w, Na, Ca, Cl\}$$

$$E = \{w, PG, Na, Ca, Cl\}$$

We deal with a three phase model.



Collagen and proteoglycans are considered as immobile, remaining within their phases, S and EF respectively.

As a measure of the chemical composition of a species k, we use:

1) The molar fraction

$$x_{kK} = \frac{N_{kK}}{N_K}$$

of species k in the phase K.

N_{kK}: number of moles of species k in the phase K

N_K: total number of moles in phase K.

2) The concentration

$$c_{kE} = \frac{N_{kE}}{V_E}$$

per phase volume V_E.

The effective corresponding quantities involve the valences, e.g., the effective concentration is

$$e_{PG} = \zeta_{PG} c_{PG} = \zeta_{PG} \frac{N_{PG}}{V_E} .$$

ζ_{PG}: the valence of the PG's.

<u>Work and electrochemical potentials</u>

$$\underbrace{d\underline{E}}_{\substack{\text{Incremental}\\ \text{work done}\\ \text{per unit}\\ \text{initial}\\ \text{volume}}} = \underbrace{\underset{\sim}{T} : d\underset{\sim}{E}}_{\substack{\text{mechanical}\\ \text{contribution}}} + \underbrace{\sum_{k,K} \mu^{ec}_{kK}\, dm^{kK}}_{\substack{\text{electrochemical}\\ \text{contribution}}}$$

$\underset{\sim}{T}$: $2^{\underline{nd}}$ order stress tensor

$\underset{\sim}{E}$: 2^{nd} strain tensor

$\underline{E}$: work done per unit volume, in the undeformed configuration

μ^{ec}_{kK}: electrochemical potential of species k, in phase K.

m^{kK}: addition of mass of species k in phase K.

Definition of the electrochemical potentials

$$g^{ec}_{kK} = \hat{m}_k \, \mu^{ec}_{kK}$$

g^{ec}_{kK}: mole based electrochemical potential

$\hat{m}_k$: molar mass

μ^{ec}_{kK}: mass based electrochemical potential

For compressible materials,

$$g^{ec}_{k\bar{K}} = \underbrace{\int \hat{v}_k \, dp_{kK}}_{\text{mechanical contribution}} + \underbrace{RT \ln x_{k\bar{K}}}_{\text{chemical contribution}} + \underbrace{z_k F \phi_k}_{\text{electrical contribution}}, \quad k \in \bar{K}$$

$\int$: integral from a reference state to the current state

$\hat{v}_k$: molar volume of species k

p_{kK}: generalised pressure of species k, belonging ~~to~~ to the phase $\bar{K}$.

R: universal gas constant

T: absolute temperature

$x_{k\bar{K}}$: molar fraction of species k in phase $\bar{K}$

z_k: valence of species k

F: Faraday's constant

ϕ_k: electrical potential, of species k.

For incompressible materials, we express the electrochemical potential via the relation

$$g_{kk}^{ec} = \hat{m}_k \mu_{kk}^{ec} = \hat{v}_k P_{kk} + RT \ln x_{kk} + \zeta_k F \phi_k$$

when the species k is electrically neutral, the valence ζ_k is equal to zero, and we talk about a chemical potential (not an electrochemical potential).

Balance equations

1) Mass balance in local form

$$\underbrace{\frac{1}{\det \underset{\sim}{F}} \frac{dm^{kk}}{dt}}_{\text{Local change of mass}} = \underbrace{\hat{\rho}^{kk}}_{\text{mass transfer}} - \underbrace{\operatorname{div} M_{kk}}_{\text{diffusion}}$$

For electrical charge, we write the balance equation as

$$\operatorname{div} I_{eE} = 0$$

where I_{eE} is the electric current density.

2) Balance of linear momentum

$$\operatorname{div} \underset{\sim}{\sigma} + \rho \vec{b} = 0$$

For incompressibility, we write

$$d\rho_k = 0$$

for species k.

If all species are incompressible, we write

$$\operatorname{div} \vec{v}_s + \operatorname{div} \vec{J}_E = 0$$

$\vec{J}_i$: fluxes in ~~and~~ out from phase E.

Chemomechanical constitutive equations for cartilage

The chemoelastic energy density $\underline{W}$ for cartilage is writen as

$$d\underline{W} = \underset{\sim}{\overline{T}} : d\underset{\sim}{E} + \sum_{(i,n)\in(Ein,\, Ene)} \overline{\mu}_{nE}\, dm^{iE} + \mu_{adh}\, dm^{wI}$$

in: independent species, like ~~ions~~ Na, Ca ..

ne: neutral elements like salts

μ_{adh}: represents the energy necessary to absorb water, from the intrafibrilar phase.

We can write the constitutive equations, as

$$\underset{\sim}{S} = \frac{\partial W}{\partial \underset{\sim}{\varepsilon}}$$

where

$\underset{\sim}{S}, \underset{\sim}{\varepsilon}$: generalised stress and strain (not actually) tensors

Explicitly,

$$\underset{\sim}{\overline{T}} = \frac{\partial \underline{W}}{\partial \underset{\sim}{E}}, \quad \overline{\mu}_{nE} = \frac{\partial W}{\partial m^{iE}}, \quad \mu_{adh} \stackrel{!}{=} \frac{\partial W}{\partial m^{wI}}$$