5. ΘΕΩΡΗΜΑ ΑΝΤΙΚΑΤΑΣΤΑΣΗΣ

5.1. Μετασχηματισμοί στο επίπεδο.

Ορισμός 5.1.1. Μια απεικόνιση $T : U \to \mathbb{R}^2$, όπου $U \subseteq \mathbb{R}^2$ ανοικτό, ονομάζεται <u>μετασχηματισμός</u>.

Εάν $T: U \to \mathbb{R}^2$ μετασχηματισμός, ορίζονται δύο συναρτήσεις $x(\cdot,\cdot), y(\cdot,\cdot): U \to \mathbb{R}$ |

$$T(u,v) = \left(x(u,v),\ y(u,v)\right),\ \forall (u,v) \in U$$

Ο T λέγεται $\underline{C^1}$ ανν οι x, y είναι C^1 δηλ.

ανν έχουν συνεχείς μερικές παραγώγους.

Ορισμός 5.1.2: Έστω $T : U \to \mathbb{R}^2$ C^1 - μετασχ.

$\forall\ \xi_0 = (u_0, v_0) \in U$, ορίζονται:

→ **ο Ιακωβιανός πίνακας του T** **στο ξ_0**:

$$T'(\xi_0) = \begin{bmatrix} x_u(\xi_0) & x_v(\xi_0) \\ y_u(\xi_0) & y_v(\xi_0) \end{bmatrix}.$$

→ **η Ιακωβιανή ορίζουσα του T στο ξ_0:**

$$J_T(\xi_0) = \det[T'(\xi_0)].$$

Παραδείγματα:

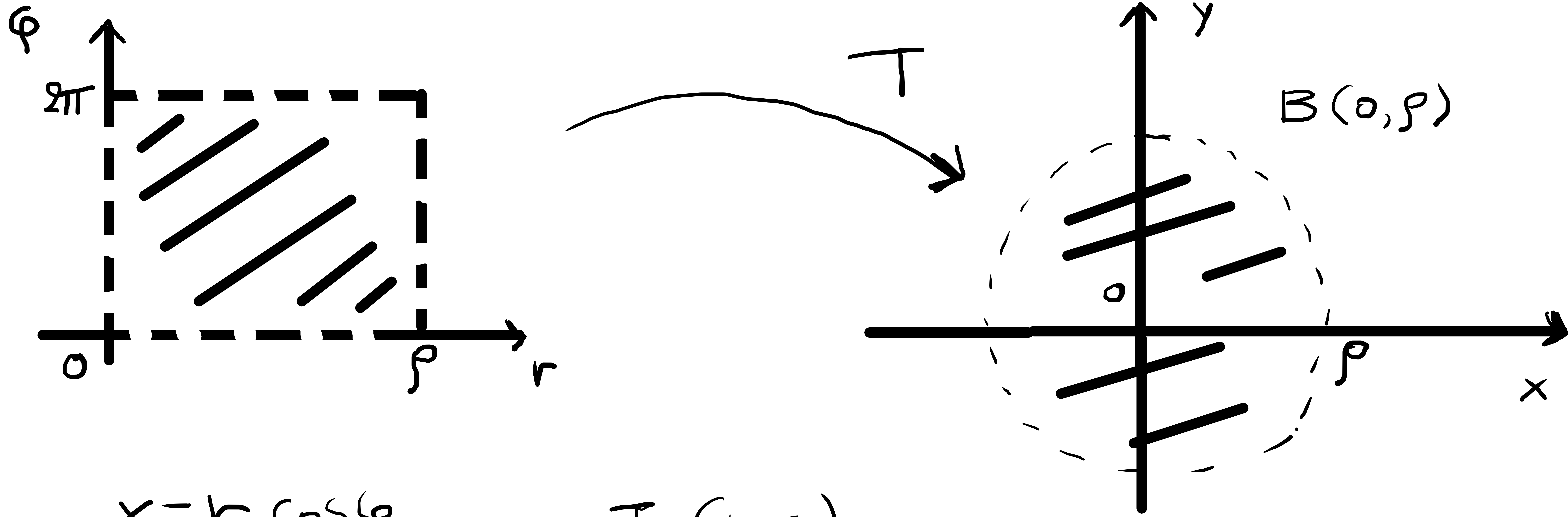
(i) Έστω M πίνακας 2×2 κ' $T:\mathbb{R}^2 \to \mathbb{R}^2$ με

$$T(u,v) = M\cdot\begin{bmatrix} u \\ v \end{bmatrix} \quad \left(\text{ταύτιση: } \begin{bmatrix} u \\ v \end{bmatrix} \equiv (u,v)\right).$$

Τότε, $J_T(u,v) = \det(M)$, $\forall (u,v)\in\mathbb{R}^2$.

(ii) (πολικές συντεταγμένες) $T:(0,\rho)\times(0,2\pi)\to\mathbb{R}^2$ $(\rho>0)$

με $T(r,\varphi) = (r\cos\varphi,\ r\sin\varphi)$, $\forall r\in(0,\rho)$, $\varphi\in(0,2\pi)$.



$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$J_T(r,\varphi) = \begin{vmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r.$$

Θεώρημα 5.1.3 (Αντίστροφης απεικόνισης).

Έστω $T: U \to \mathbb{R}^2$ C^1- μετασχ. κ' $J_T(\xi_0) \neq 0$, $\xi_0 \in U$.

Τότε, $\exists$ W ανοικτό ώστε

- $\xi_0 \in W \subseteq U$
- $T|_W$ 1-1
- $T(W)$ ανοικτό
- ο $T^{-1} : T(W) \to W$ είναι C^1.

κ'

$$J_{T^{-1}}(T(\xi_0)) = \frac{1}{J_T(\xi_0)}.$$

Πόρισμα 5.1.4: Έστω $T: U \to \mathbb{R}^2$ **C^1-μετασχ.**, **1-1**, με $J_T(\xi) \neq 0$, $\forall \xi \in U$. Τότε, $T(U)$ ανοικτό, η $T^{-1}: T(U) \to U$ είναι C^1 και

$$J_{T^{-1}}(T(\xi)) = \frac{1}{J_T(\xi)}, \quad \forall \xi \in U.$$

Σχόλιο: Εάν $J_T(\xi) \neq 0$, $\forall \xi \in U$, <u>δεν</u> έπεται ότι T 1-1.

π.χ. $T(u,v) = (e^u \cos v, e^u \sin v)$, $(u,v) \in \mathbb{R}^2$.

5.2. Θεώρημα Αντικατάστασης.

Έστω $U \subseteq \mathbb{R}^2$ ανοικτό κ' $T: U \to \mathbb{R}$ C^1-μετασχ. ώστε

$$T \ 1-1, \quad J_T(\xi) \neq 0, \ \forall \xi \in U.$$

Έστω επιπλέον $A \subset U$ κλειστό, φραγμένο, Jordan μετρήσιμο. Τότε:

(i) $T(A)$ κλειστό, φραγμένο κ' Jordan μετρήσιμο.

(ii) Εάν $f: T(A) \to \mathbb{R}$ συνεχής, ισχύει

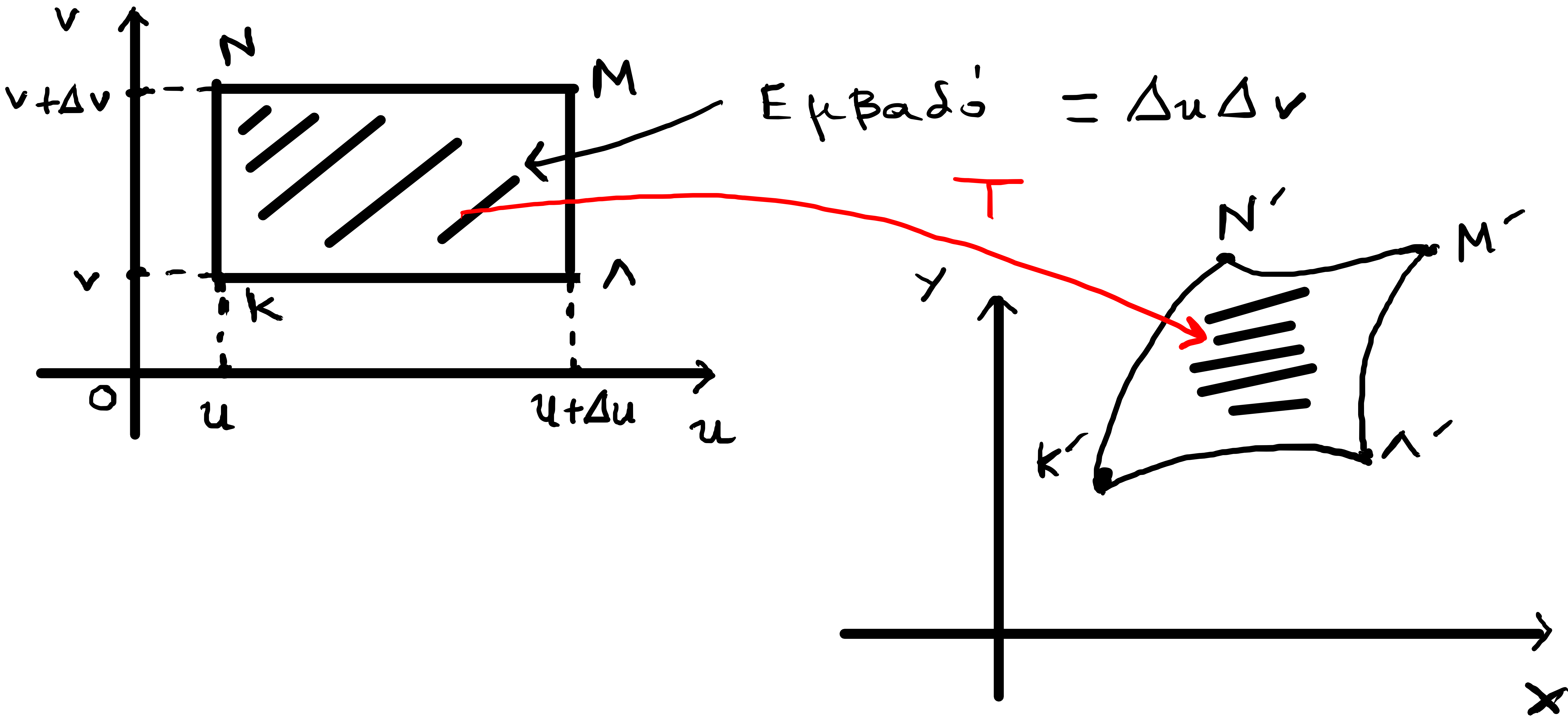
$$\iint_A f(T(u,v))\,|J_T(u,v)|\,du\,dv = \iint_{T(A)=D} f(x,y)\,dx\,dy.$$

$$T:\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$J_T = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

<u>Μια "διαισθητική" απόδειξη για $f = 1$.</u>

$$A = T^{-1}(D)$$


v
N
M
v+Δv
Εμβαδό = ΔuΔv
T
N´
M´
v
Λ
y
K
O
u
u+Δu
u
K´
Λ´
x

- $K(u,v) \xmapsto{T} K'(x(u,v), y(u,v))$
- $\Lambda(u+\Delta u, v) \xmapsto{T} \Lambda'(x(u+\Delta u, v), y(u+\Delta u, v))$
- $N(u, v+\Delta v) \xmapsto{T} N'(x(u, v+\Delta v), y(u, v+\Delta v))$

$$\overrightarrow{K'\Lambda'} = (x(u+\Delta u, v) - x(u,v),\ y(u+\Delta u, v) - y(u,v))$$

$$\cong (x_u \Delta u,\ y_u \Delta u) = \Delta u (x_u, y_u)$$

όμοια,

$$\overrightarrow{K'N'} \cong \Delta v (x_v, y_v).$$

Εμβαδό καμπυλόγραμμου χωρίου $T((K\Lambda MN)) \cong$

$\cong$ Εμβαδό του $\#(K'\Lambda'M'N') =$

$$= \left| \begin{vmatrix} x_u \Delta u & y_u \Delta u \\ x_v \Delta v & y_v \Delta v \end{vmatrix} \right| = \Delta u \Delta v \, |J_T(u,v)|$$

δηλ. $\Delta x \Delta y \cong \Delta u \Delta v \, |J_T(u, v)|.$

<u>Σχόλιο:</u> Εάν $T : [0, 2\pi] \times [0, \rho] \to \mathbb{R}^2$ $(\rho > 0)$

με $T(r, \varphi) = (r\cos\varphi, r\sin\varphi)$, ο T <u>δεν</u> είναι 1-1

αφού $T(r, 0) = T(r, 2\pi)$, $\forall r > 0$ κ' το

$[0, 2\pi] \times [0, \rho]$ δεν είναι ανοικτό.

Ο $T|_U : U = (0, 2\pi) \times (0, \rho) \to \mathbb{R}^2$ είναι 1-1.

Έστω $0 < \delta < \rho/2$, $0 < \varepsilon < \pi$,

$$A(\delta, \varepsilon) = [\delta, \rho - \delta] \times [\varepsilon, 2\pi - \varepsilon] \subset U.$$

Μπορούμε να εφαρμόσουμε το Θ. Αντικατάστασης στο $A(\delta,\varepsilon)$ κ' παίρνουμε

$$\iint_{A(\delta,\varepsilon)} f(r\cos\varphi, r\sin\varphi)\, r\, dr\, d\varphi = \iint_{T(A(\delta,\varepsilon))} f(x,y)\, dx\, dy$$

$$\xRightarrow{\delta,\varepsilon \downarrow 0^+} \boxed{\iint_{[0,\rho]\times[0,2\pi]} f(r\cos\varphi, r\sin\varphi)\, r\, dr\, d\varphi = \iint_{B[0,\rho]} f(x,y)\, dx\, dy.}$$

Παραδείγματα:

(i) $I = \iint_D e^{\frac{x-y}{x+y}}\, dx\,dy = ?$ $\quad D = \left\{ (x,y) \,\middle|\, x+y \le 1,\ \begin{matrix} x \geq 0 \\ y \geq 0 \end{matrix} \right\}$

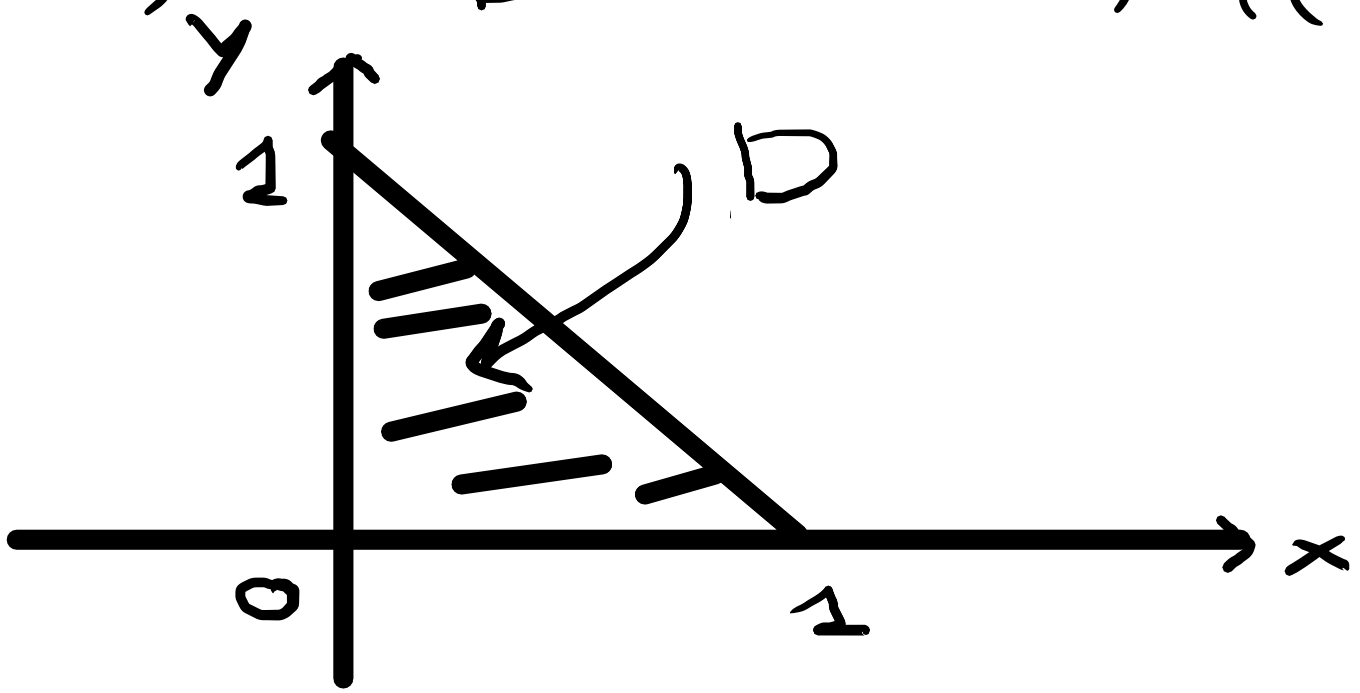
Θέτουμε $\left.\begin{matrix} u = x-y \\ v = x+y \end{matrix}\right\} \Leftrightarrow \begin{matrix} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{matrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(u,v) = \left(\frac{u+v}{2}, \frac{v-u}{2} \right)$

T C^1, 1−1 κ' $J_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} =$

$= \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/2 \neq 0, \quad \forall (u,v).$

Επίσης, D κλειστό, φραγμένο, Jordan μετρήσιμο.



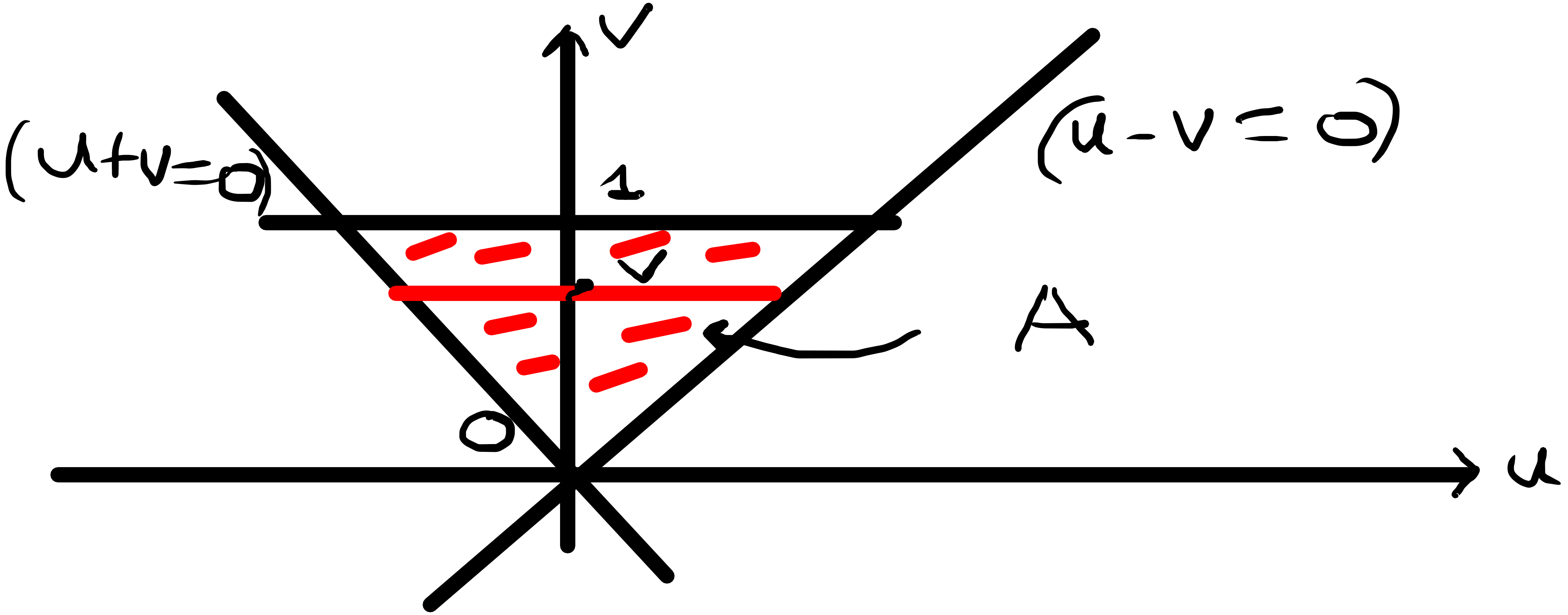
Θέτω

$A = T^{-1}(D)$

$\Leftrightarrow T(A) = D.$

$$A = T^{-1}(D) = ?$$

$$\left.\begin{array}{l} x+y \le 1 \\ x \geq 0 \\ y \geq 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} v \le 1 \\ u+v \geq 0 \\ v-u \geq 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} v \le 1 \\ u+v \geq 0 \\ u-v \le 0 \end{array}\right\}$$



Για $v \in [0,1]$, έχουμε

$$-v \le u \le v.$$

$$I = \iint_A e^{u/v} \frac{1}{2}\, du\, dv = \frac{1}{2}\int_0^1 \left(\int_{-v}^{v} e^{u/v}\, du \right) dv$$

$$= \frac{1}{2} \int_0^1 v e^{u/v} \Big|_{u=-v}^{u=v} dv = \frac{1}{2}\left(e - e^{-1}\right) \int_0^1 v\, dv$$

$$= \frac{1}{4}\left(e - \frac{1}{e}\right).$$

(ii) $I = \iint_D (x^4 - y^4)\, e^{xy}\, dx\, dy = ?$

$D = \left\{ (x,y) \,\middle|\, 1 \leq xy \leq 2, \quad 2 \leq x^2 - y^2 \leq 3, \quad x \geq 0, \ y \geq 0 \right\}$

Λύση: καταρχήν παρατηρούμε ότι $\forall (x,y) \in D$, $x > 0$, $y > 0$.

Θέτουμε $u = xy$, $v = x^2 - y^2$, δηλ. θεωρούμε

το μετασχ.

$$S : D \to \mathbb{R}^2, \quad S(x,y) = (xy,\ x^2 - y^2).$$

Προφανώς S C^1-μετασχ.

• **S 1-1** Πράγματι έστω $(x_1, y_1), (x_2, y_2) \in D$

με $\begin{cases} x_1 y_1 = x_2 y_2 \\ x_1^2 - y_1^2 = x_2^2 - y_2^2 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x_1}{x_2} = \dfrac{y_2}{y_1} = \lambda > 0 \\ x_1^2 - y_1^2 = x_2^2 - y_2^2 \end{cases}$

$$\Rightarrow \lambda^2 x_2^2 - y_1^2 = x_2^2 - \lambda^2 y_1^2 \Leftrightarrow (\lambda^2 - 1)(x_2^2 + y_1^2) = 0$$

$\Leftrightarrow$ $\lambda = 1$ οπότε $x_1 = x_2$, $y_1 = y_2$.

• $\forall (x,y) \in D$, $J_S(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} =$

$= -2y^2 - 2x^2 \neq 0$

• $S(D) = [1,2] \times [2,3]$. Πράγματι:

$D = \{(x,y) \mid 1 \leq xy \leq 2,\ 2 \leq x^2 - y^2 \leq 3\}$

$S(x,y) = (u,v) = (xy,\ x^2 - y^2)$.

Προφανώς, $S(D) \subseteq [1,2] \times [2,3]$.

- Έστω $(u,v) \in [1,2] \times [2,3]$. Θα δ.ο. $\exists\, x,y>0 \mid$

$$\left.\begin{array}{l} xy = u \\ x^2 - y^2 = v \end{array}\right\} \Leftrightarrow \left\{\begin{array}{ll} y = u/x & (1) \\ x^4 - vx^2 - u^2 = 0 & (2) \end{array}\right.$$

$$(2) \xRightarrow{z = x^2 > 0} z^2 - vz - u^2 = 0 \quad (3)$$

$\Delta = v^2 + 4u^2 > 0$, ενώ $-u^2 < 0$, οπότε η (3) έχει θετική ρίζα $z \Rightarrow$ η (2) έχει θετική ρίζα

κ.λ.π.

Ορίζεται ο $T = S^{-1} : \underbrace{[1,2]\times[2,3]}_{A} \to D$,

T C^1-μετασχ. 1-1 με $T(A) = D$ κ'

$\forall (u,v) \in A$, με $u = xy$, $v = x^2 - y^2$ | $J_T(u,v) = \dfrac{1}{J_S(x,y)} = -\dfrac{1}{2(x^2+y^2)}$

$$\Longrightarrow \underline{(x^2+y^2)\,|J_T| = 1/2} \quad (3)$$

Έχουμε

$$I = \iint_D (x^4 - y^4) e^{xy} \, dx\,dy = \iint_{T(A)} (x^2 - y^2) e^{xy} (x^2 + y^2) \, dx\,dy$$

$$= \iint_A u e^v (x^2 + y^2) |J_T| \, du\,dv \overset{(3)}{=} \frac{1}{2} \iint_A u e^v \, du\,dv$$

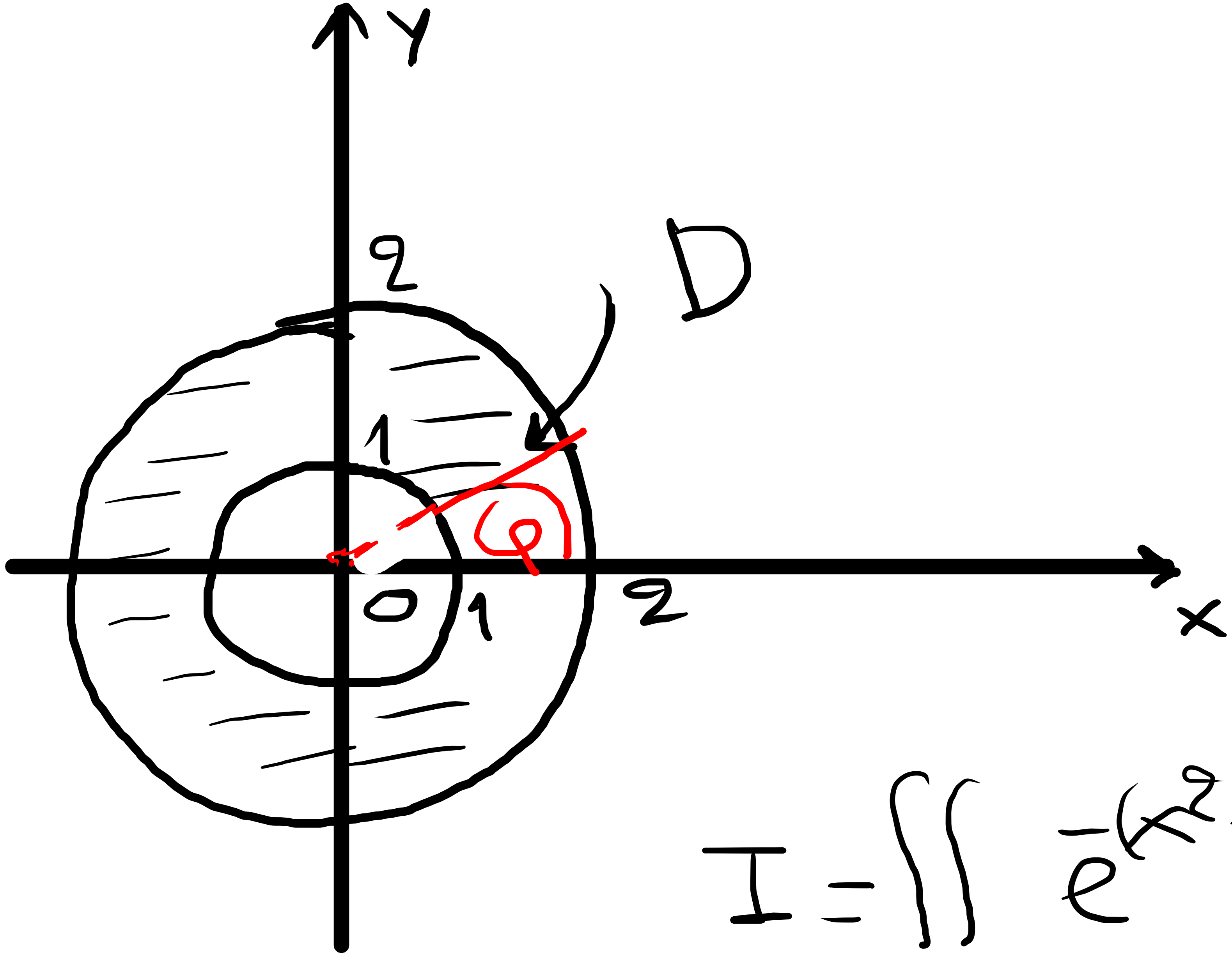
$$= \frac{1}{2} \int_1^2 u\,du \cdot \int_2^3 e^v \, dv = \frac{e^3 - e^2}{4}.$$

<u>(iii)</u> $I = \iint\limits_{D} e^{-(x^2+y^2)}\,dxdy = ?$

$D = \{ (x,y) \mid 1 \leq x^2+y^2 \leq 4 \}$.

<u>Λύση:</u> $x = r\cos\varphi$, $y = r\sin\varphi$, $x^2+y^2 = r$

$1 \leq x^2+y^2 \leq 4 \iff \boxed{1 \leq r \leq 2}$ κ' $\varphi \in [0, 2\pi]$



$$T : [1,2] \times [0,2\pi] \to D$$

$$T(r,\varphi) = (r\cos\varphi, r\sin\varphi)$$

$$|J_T| = r$$

$$I = \iint_D e^{-(x^2+y^2)}\,dx\,dy = \iint_{[1,2]\times[0,2\pi]} e^{-r^2}\, r\,dr\,d\varphi$$

$$=$$

$$= \int_0^{2\pi} d\varphi \cdot \int_1^2 e^{-r^2}\, dr = 2\pi \left(-\frac{1}{2}\right) e^{-r^2} \Big|_{r=1}^{r=2} =$$

$$= \pi \left(\frac{1}{e} - \frac{1}{e^4} \right).$$

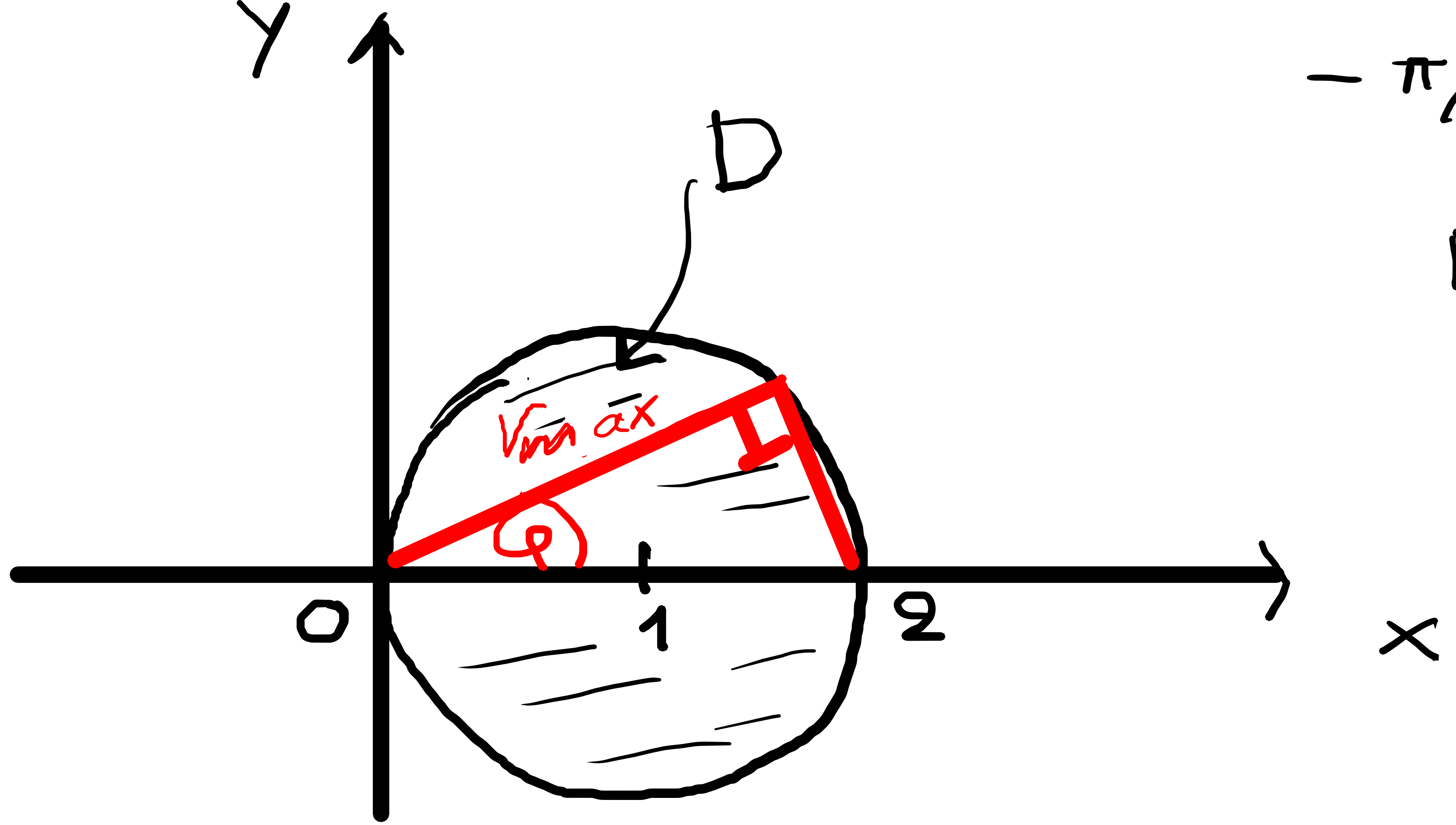
(iv) $I = \iint\limits_D \frac{dx\,dy}{\sqrt{x^2+y^2}} = ?$, $D = \{(x,y) \mid x^2+y^2 \le 2x\}$.

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ r &> 0 \end{aligned}\right\} \xRightarrow{x^2+y^2 \le 2x} \quad r^2 \le 2r\cos\varphi$$

$$\Longrightarrow \quad 0 < r \le 2\cos\varphi$$

φ?

$$x^2+y^2 \le 2x \iff (x-1)^2+y^2 \le 1$$



$-\pi/2 < \varphi < \pi/2$

Για δοσμένο

$\varphi \in (-\pi/2, \pi/2)$,

έχουμε

$0 < r \leq 2\cos\varphi$.

Οπότε $\displaystyle\iint_D \frac{dx\,dy}{\sqrt{x^2+y^2}} = \iint_{T^{-1}(D)=A} \frac{r}{r}\,dr\,d\varphi =$

$$= \iint_{T^{-1}(D)=A} dr\, d\varphi = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\cos\varphi} dr \right) d\varphi =$$

$$= 2\int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi = 4\int_0^{\pi/2} \cos\varphi\, d\varphi = 4\sin\varphi \Big|_0^{\pi/2} = 4.$$

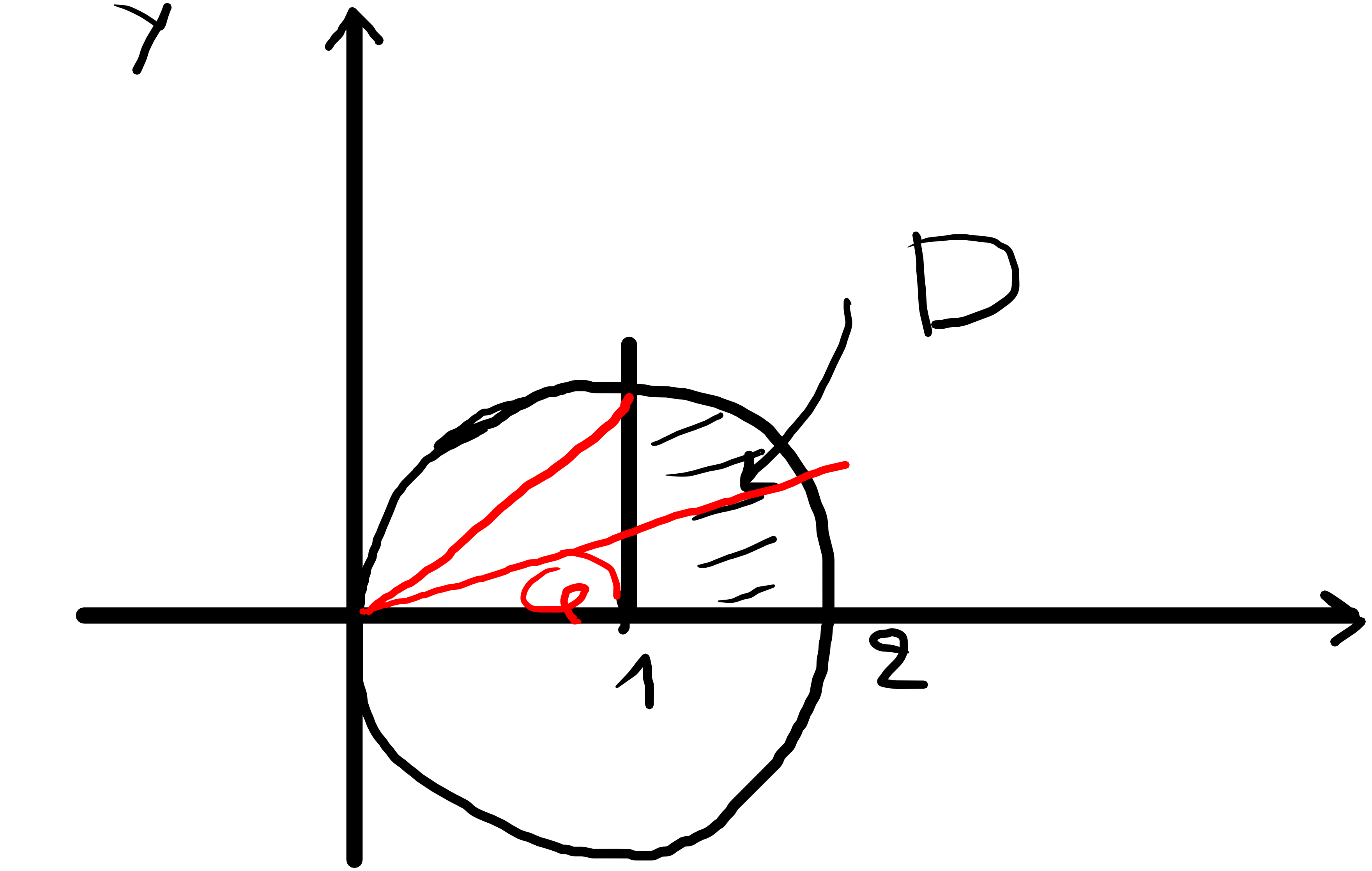
(v) $I = \iint_D \frac{x}{x^2+y^2}\,dxdy = ?$

$D = \{(x,y) \mid x^2+y^2 \leq 2x,\quad x \geqslant 1,\quad y \geqslant 0\}$

Λύση: $x = r\cos\varphi,\quad y = r\sin\varphi$

$x^2+y^2 \leq 2x \implies r \leq 2\cos\varphi$

$x \geqslant 1 \implies r \geqslant \frac{1}{\cos\varphi}$ $\qquad \varphi ?$



x

$0 \leq \varphi \leq \pi/4$

$$I = \iint\limits_{T^{-1}(D)=A} \frac{r\cos\varphi}{r^2}\, r\,dr\,d\varphi = \iint\limits_{T^{-1}(D)=A} \cos\varphi\,dr\,d\varphi =$$

$$= \int_0^{\pi/4} \cos\varphi \left(\int_{1/\cos\varphi}^{2\cos\varphi} dr \right) d\varphi = \int_0^{\pi/4} \left(2\cos^2\varphi - 1\right) d\varphi$$

$$= \int_0^{\pi/4} \cos(2\varphi)\, d\varphi = \frac{1}{2}\sin(2\varphi)\Big|_0^{\pi/4} = 1/2 .$$

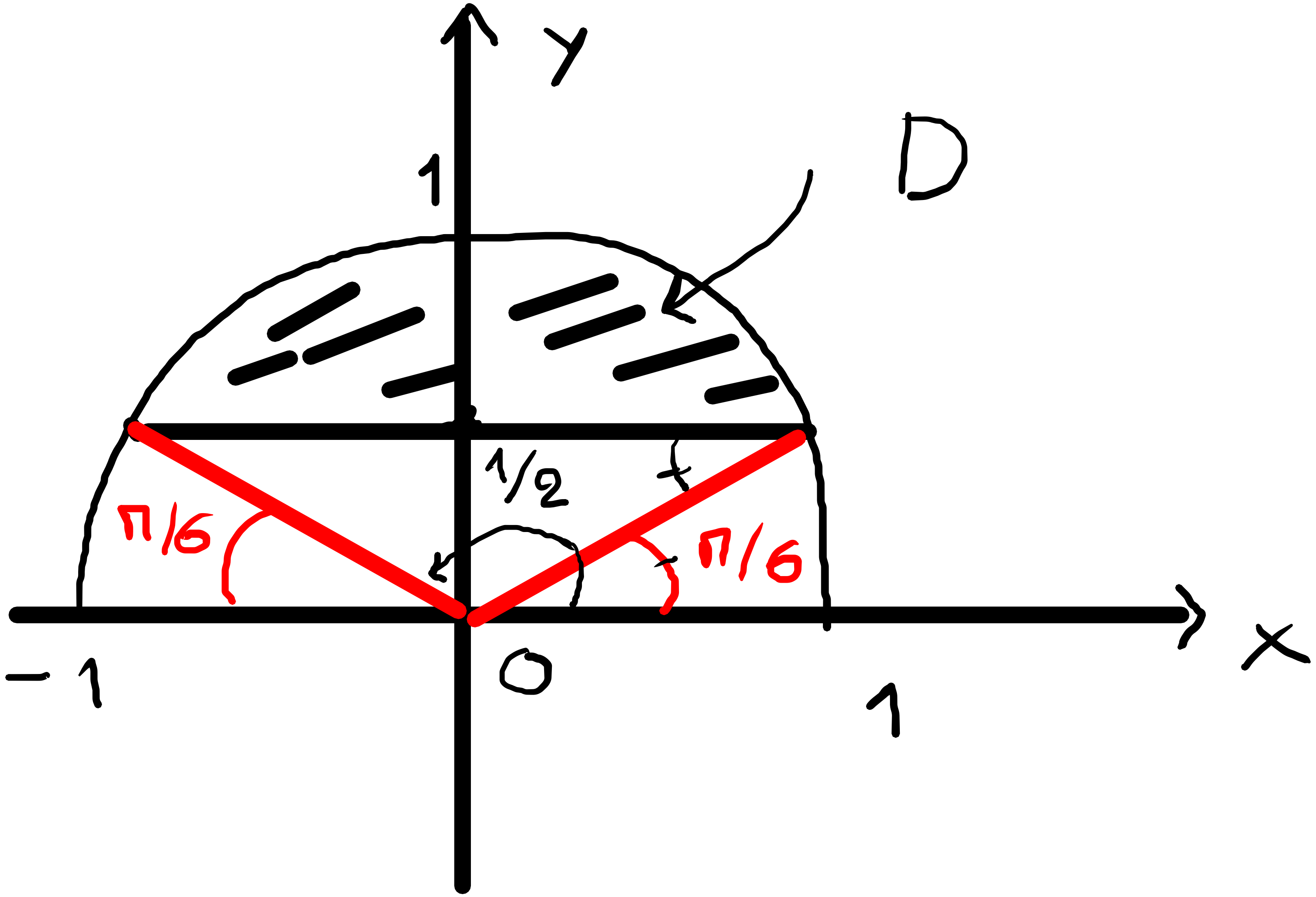
(vi) $I = \iint_D \dfrac{y^3}{(x^2+y^2)^{3/2}}\, dx\,dy = ?$

$$D = \left\{ (x,y) \mid x^2+y^2 \le 1, \quad y \geq 1/2 \right\}$$

λύση:

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \Longrightarrow \begin{aligned} &x^2+y^2 \le 1 \\ &y \geq 1/2 \end{aligned}$$

$$\Longrightarrow \boxed{\frac{1}{2\sin\varphi} \le r \le 1} \qquad \varphi\,?$$



$$\boxed{\pi/6 \leq \varphi \leq \pi - \pi/6 = 5\pi/6}$$

Εάν

$$T(r,\varphi) = (r\cos\varphi,\ r\sin\varphi)$$

κ' $A = T^{-1}(D)$,

τότε $A = \left\{ (r,\varphi) \mid \pi/6 \leq \varphi \leq 5\pi/6,\ \frac{1}{2\sin\varphi} \leq r \leq 1 \right\}$

$\Longrightarrow$

$$\Rightarrow \iint_D \frac{y^3}{(x^2+y^2)^{3/2}}\,dx\,dy = \iint_A \frac{r^3 \sin^3\varphi}{(r^2)^{3/2}}\, r\,dr\,d\varphi$$

$$= \iint_A r \sin^3\varphi \, d\varphi = \int_{\pi/6}^{5\pi/6} \sin^3\varphi \left(\int_{\frac{1}{2\sin\varphi}}^{1} r\,dr \right) d\varphi$$

$$= \frac{1}{2} \int_{\pi/6}^{5\pi/6} \sin^3\varphi \left(1 - \frac{1}{4\sin^2\varphi}\right) d\varphi =$$

$$= \underbrace{\frac{1}{2}\int_{\pi/6}^{5\pi/6} \sin^3\varphi\, d\varphi}_{I_1} - \underbrace{\frac{1}{8}\int_{\pi/6}^{5\pi/6} \sin\varphi\, d\varphi}_{I_2}$$

$$I_2 = \frac{1}{8}\cos\varphi\Big|_{\pi/6}^{5\pi/6} = \dots$$

$$I_1 = -\int_{\pi/6}^{5\pi/6} (1-\cos^2\varphi)\, d(\cos\varphi) \overset{u=\cos\varphi}{=} -\int_{\sqrt{3}/2}^{-\sqrt{3}/2} (1-u^2)\, du$$

$$= 2\int_0^{\sqrt{3}/2} (1-u^2)\, du = \dots$$

(vii) $I = \iint_D \operatorname{Arctan}\left(\frac{y}{x}\right) dxdy = ?$

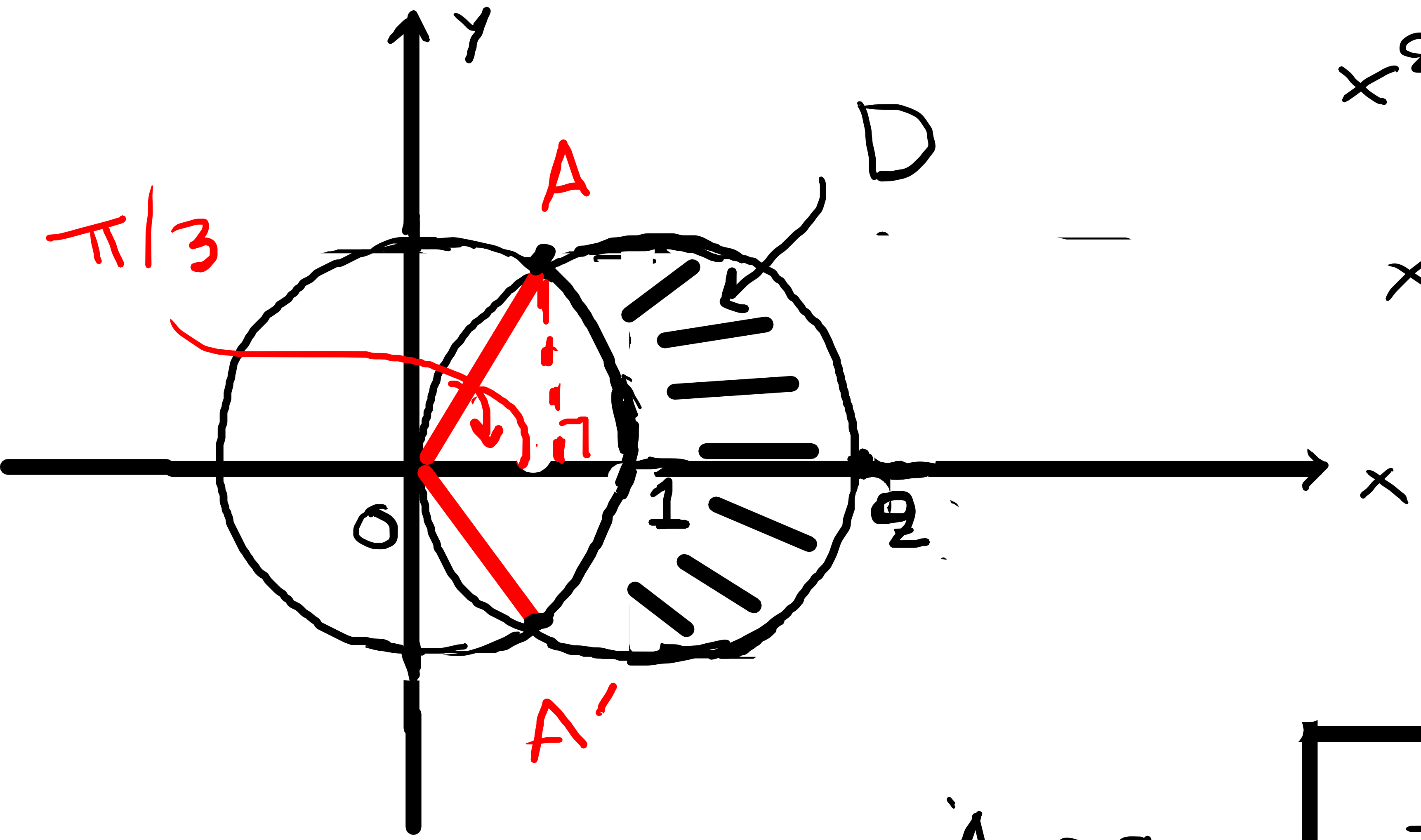
$D = \{(x,y) \mid 1 \leq x^2+y^2 \leq 2x\}$

Λύση:

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \xRightarrow{(x,y)\in D} 1 \leq r^2 \leq 2r\cos\varphi$$

$$\Rightarrow \boxed{1 \leq r \leq 2\cos\varphi}$$

φ ?



$$\left.\begin{aligned} x^2+y^2&=1 \\ x^2+y^2&=2x \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} x&=1/2 \\ y&=\pm\sqrt{3}/2 \end{aligned}\right.$$

$A\left(1/2,\ \sqrt{3}/2\right)$

$A'\left(1/2, -\sqrt{3}/2\right)$

Άρα, $\boxed{-\pi/3 \le \varphi \le \pi/3}$

Εάν $T(r,\varphi) = (r\cos\varphi,\ r\sin\varphi)$, τότε

$$T^{-1}(D) = A = \{(r,\varphi) \mid -\pi/3 \leq \varphi \leq \pi/3,\ 1 \leq r \leq 2\cos\varphi\}$$

$$\Rightarrow \iint_D \operatorname{Arctan}\left(\frac{y}{x}\right) dx\,dy = \iint_A \operatorname{Arctan}\left(\frac{r\sin\varphi}{r\cos\varphi}\right) r\,dr\,d\varphi$$

$$= \iint_A \varphi \cdot r\,dr\,d\varphi \quad =$$

$$= \int_{-\pi/3}^{\pi/3} \varphi \left(\int_{1}^{2\cos\varphi} r\,dr \right) d\varphi = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \underbrace{\varphi\left(4\cos^2\varphi - 1\right)}_{(\text{περιττή})} d\varphi$$

$$= 0.$$

$\underline{(viii)}$ $I = \iint_D \dfrac{x+y}{x^2+y^2}\,dx\,dy = ?$

$$D = \left\{ (x,y) \;\middle|\; 1 \le x^2+y^2, \quad x^2 + \frac{y^2}{4} \le 1 \right\}$$

<u>Λύση:</u>

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\} \xRightarrow{(x,y)\in D}$$

$\underline{1 \le r,}$ $\quad 4r^2\cos^2\varphi + r^2\sin^2\varphi \le 4$ ή

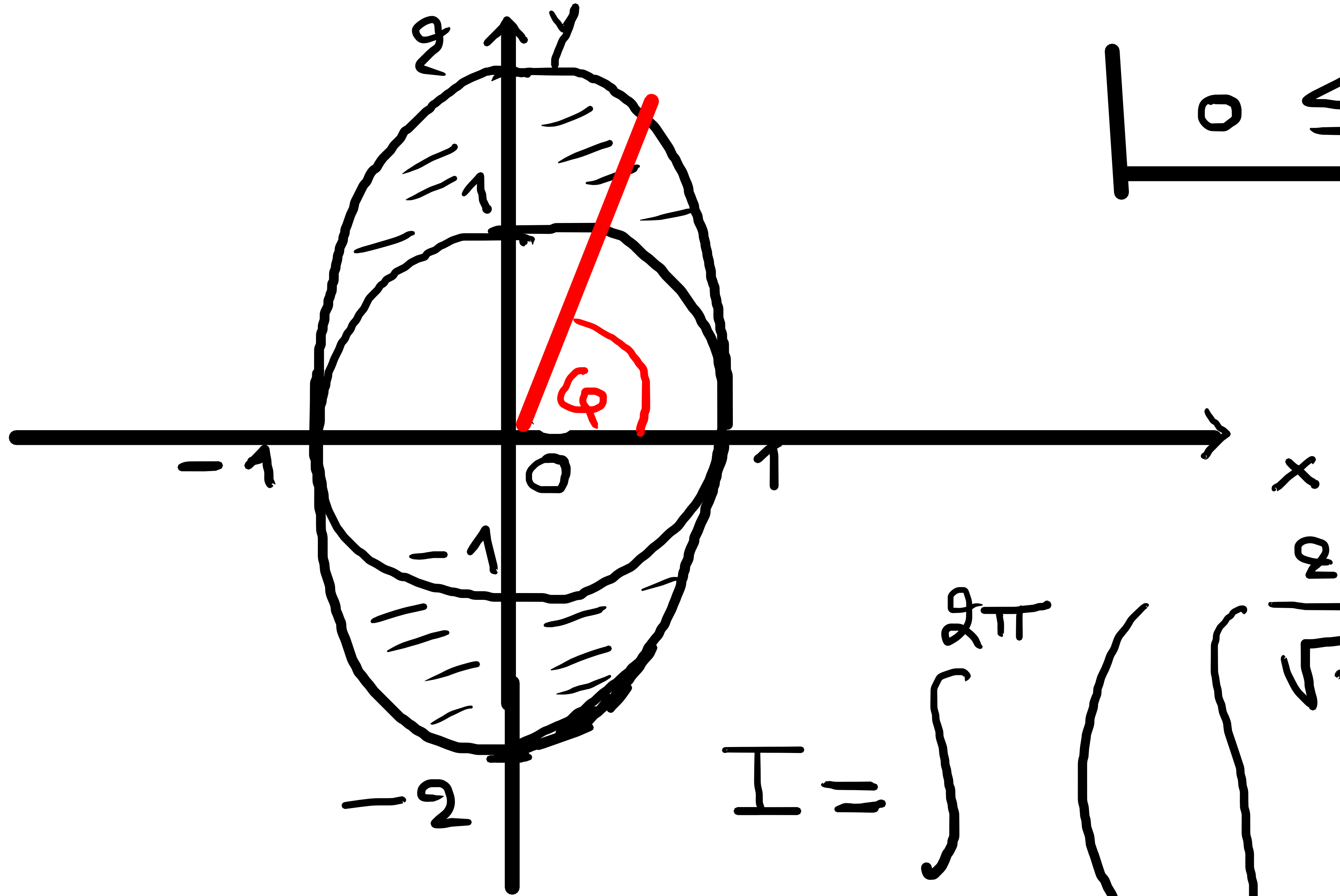
$$r^2(4\cos^2\varphi + \sin^2\varphi) \leq 4$$

$$\Rightarrow \quad r^2(1+3\cos^2\varphi) \leq 4 \quad \Longrightarrow \quad r \leq \frac{2}{\sqrt{1+3\cos^2\varphi}}.$$

Τελικά,

$$\boxed{1 \leq r \leq \frac{2}{\sqrt{1+3\cos^2\varphi}}}$$

φ ?



$$\boxed{0 \le \varphi \le 2\pi}$$

$$I = \int_0^{2\pi} \left(\int_1^{\frac{2}{\sqrt{1+3\cos^2\varphi}}} \frac{r(\cos\varphi + \sin\varphi)}{r^2}\, r\, dr \right) d\varphi$$

$$=$$

$$= \int_0^{2\pi} \underbrace{(\cos\varphi + \sin\varphi)\left(\frac{2}{\sqrt{1+3\cos^2\varphi}} - 1\right)}_{g(\varphi)} d\varphi$$

$$g(\varphi + \pi) = -g(\varphi)$$

$$I = \int_0^{\pi} g(\varphi)\, d\varphi + \int_{\pi}^{2\pi} g(\varphi)\, d\varphi .$$

$$\int_{\pi}^{2\pi} g(\varphi)\,d\varphi \overset{\theta=\varphi-\pi}{=} \int_0^{\pi} g(\theta+\pi)\,d\theta = -\int_0^{\pi} g(\theta)\,d\theta$$

$$\Longrightarrow \qquad I = 0.$$

(ix) Θέτουμε $I_R = \int_0^R e^{-x^2} dx$, $R > 0$.

(α) Να δ.ο. $I_R^2 = \iint_{T_R} e^{-(x^2+y^2)} dx dy$, όπου

$T_R = [0, R] \times [0, R]$.

(β) Να υπολογίσετε το $\int_0^{+\infty} e^{-x^2} dx$.

Λύση: (α) $I_R^2 = \int_0^R e^{-x^2}dx \cdot \int_0^R e^{-x^2}dx =$

$= \int_0^R e^{-x^2}dx \cdot \int_0^R e^{-y^2}dy =$

$= \int_0^R e^{-y^2}\left(\int_0^R e^{-x^2}dx\right)dy \overset{\text{(Fubini)}}{=} \iint_{[0,R]\times[0,R]} e^{-x^2}\cdot e^{-y^2}\,dx\,dy$

$= \iint_{T_R} e^{-(x^2+y^2)}\,dx\,dy.$

$(*)$

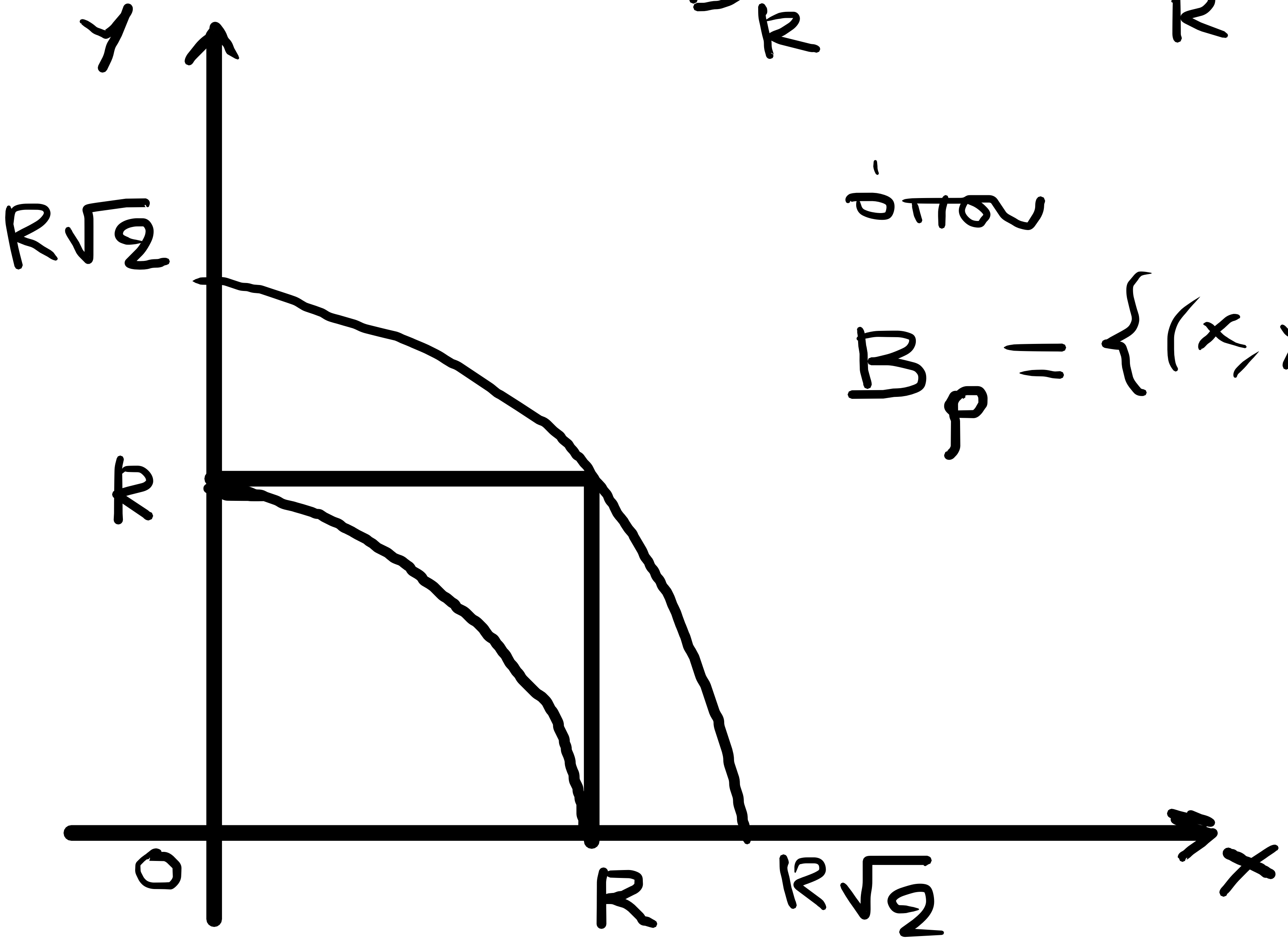
(*) Γενικά,

$$\int_a^b \varphi(x)\,dx \cdot \int_c^d g(x)\,dx =$$

$$= \iint_{[a,b]\times[c,d]} \varphi(x)g(y)\,dx\,dy$$

(β)

$$B_R \subset T_R \subset B_{R\sqrt{2}}$$



όπου

$$B_\rho = \{(x,y) \mid x^2+y^2 \leq \rho^2, \; x \geq 0, \; y \geq 0\}$$

$\rho \in \{R, R\sqrt{2}\}$.

Επομένως,

$$\iint_{B_R} e^{-(x^2+y^2)}\,dx\,dy \le \iint_{T_R} e^{-(x^2+y^2)}\,dx\,dy = I_R^2 \le$$

$$\le \iint_{B_{R\sqrt{2}}} e^{-(x^2+y^2)}\,dx\,dy.$$

Αλλά, $\forall \rho > 0$,

$$\iint_{B_\rho} e^{-(x^2+y^2)}\,dx\,dy \underset{y=r\sin\varphi}{\overset{x=r\cos\varphi}{=\!=}} \int_0^{\pi/2}\left(\int_0^{\rho} r e^{-r^2}\,dr\right)d\varphi$$

$$= \frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)e^{-r^2}\Big|_0^{\rho} = \frac{\pi}{4}\left(1-e^{-\rho^2}\right)$$

$$\Longrightarrow$$

$$\Rightarrow \iint_{B_R} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}\left(1-e^{-R^2}\right),$$

$$\iint_{B_{R\sqrt{2}}} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}\left(1-e^{-2R^2}\right)$$

$$\Rightarrow \quad \frac{\pi}{4}\left(1-e^{-R^2}\right) \leq I_R^2 \leq \frac{\pi}{4}\left(1-e^{-2R^2}\right).$$

Παίρνοντας το όριο καθώς $R \to +\infty$,

από το κριτήριο παρεμβολής παίρνουμε

$$\lim_{R\to+\infty} I_R^2 = \pi/4$$

$$\Rightarrow \quad \lim_{R\to+\infty} I_R = \sqrt{\pi}/2 \quad \Rightarrow \quad \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

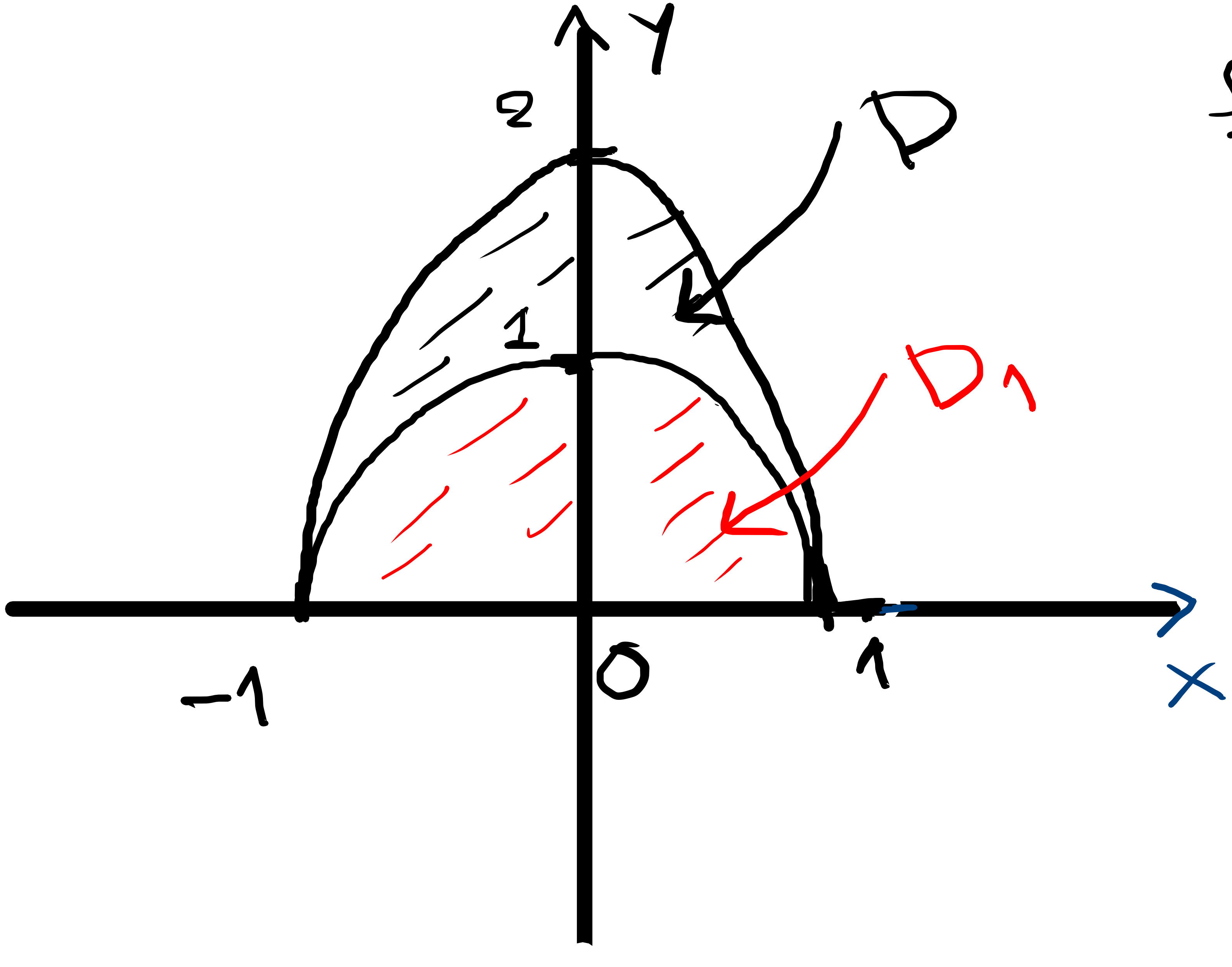
(X) $I = \iint_D (x-2y)\,dx\,dy = ?$

$D = \left\{ (x,y) \mid 1 \le x^2+y^2,\ \ x^2+\frac{y^2}{4} \le 1,\ y \ge 0 \right\}$

Λύση: $D_1 = \left\{ (x,y) \mid x^2+y^2 \le 1,\ y \geq 0 \right\}$

$D_2 = \left\{ (x,y) \mid x^2+\frac{y^2}{4} \le 1,\ y \geq 0 \right\}$

$D_2 = D \cup D_1, \qquad \overset{\circ}{D} \cap \overset{\circ}{D_1} = \emptyset$



$$f(x,y) = x - 2y$$

$$\iint_{D_2} f = \iint_{D} f + \iint_{D_1} f$$

$$\Rightarrow \underbrace{\iint_{D} f}_{I} = \underbrace{\iint_{D_2} f}_{I_2} - \underbrace{\iint_{D_1} f}_{I_1}$$

• Υπολογισμός του $I_2 = \iint_{D_2} (x-2y)\,dx\,dy$,

$$D_2 = \left\{ (x, y) \mid x^2 + \frac{y^2}{4} \le 1, \; y \geq 0 \right\}.$$

Θέτουμε $\begin{cases} x = r\cos\varphi \\ y = 2r\sin\varphi \end{cases} \xRightarrow{(x,y)\in D_2} \boxed{0 \le r \le 1}$

$$J_T = \begin{vmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ 2\sin\varphi & 2r\cos\varphi \end{vmatrix} = 2r$$

$$\Rightarrow \quad I_2 = \int_0^{\pi} \left(\int_0^1 (r\cos\varphi - 4r\sin\varphi)\, r\, dr \right) d\varphi$$

$$= \int_0^{\pi} (\cos\varphi - 4\sin\varphi)\, d\varphi \int_0^1 r^2\, dr$$

$$= \left(\sin\varphi \Big|_0^{\pi} + 4\cos\varphi \Big|_0^{\pi} \right) \cdot \frac{1}{3} = -8/3$$

• Υπολογισμός του $I_1 = \iint_{D_1} (x-2y)\,dx\,dy$,

$D_1 = \{(x,y) \mid x^2+y^2 \leq 1,\ y \geq 0\}$.

Περνάμε σε πολικές,

$$I_1 = \int_0^{\pi} \left(\int_0^1 (r\cos\varphi - 2r\sin\varphi)\, r\,dr \right) d\varphi$$

$$= \int_0^{\pi} (\cos\varphi - 2\sin\varphi)\,d\varphi \cdot \int_0^1 r^2\,dr =$$

$$= \frac{1}{3} \int_0^{\pi} (\cos\varphi - 2\sin\varphi)\, d\varphi = \frac{1}{3} \left(\sin\varphi \Big|_0^{\pi} + 2\cos\varphi \Big|_0^{\pi} \right)$$

$$= \frac{1}{3}(-4) = -4/3.$$

Άρα, $I = I_2 - I_1 = -8/3 + 4/3 = -4/3.$

(xi) $\iint_D (x+y)\,dx\,dy = ?$ $D = \{(x,y) \mid x^2+2y^2-2x-8y+5 \le 0\}$.

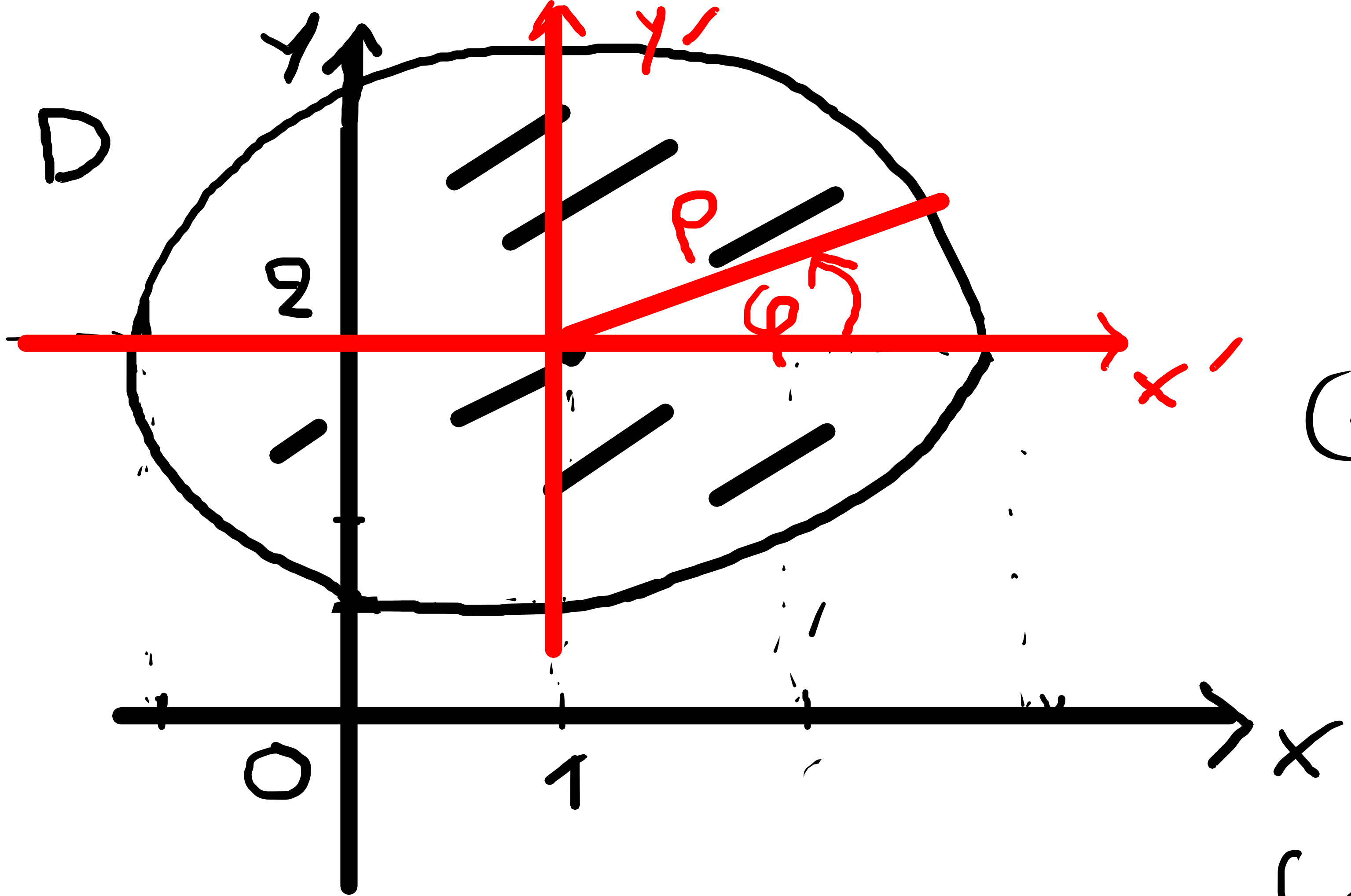
Λύση:

$$(x,y) \in D \iff x^2-2x+2(y^2-4y)+5 \le 0$$

$$\iff (x-1)^2-1+2\left[(y-2)^2-4\right]+5 \le 0$$

$$\iff (x-1)^2+2(y-2)^2 \le 4$$

$$\iff \frac{(x-1)^2}{2^2}+\frac{(y-2)^2}{\sqrt{2}^2} \le 1$$



Θέτουμε

$x = 1 + 2\rho\cos\varphi, \quad y = 2 + \sqrt{2}\rho\sin\varphi$

$(x,y) \in D \Longrightarrow \rho^2 \le 1 \Leftrightarrow \underline{0 \le \rho \le 1}.$

κ' $\underline{0 \le \varphi \le 2\pi}.$

$A = \{(\rho,\varphi) \mid 0 \le \rho \le 1, \ 0 \le \varphi \le 2\pi\}$

$T : A \to D, \quad T(\rho,\varphi) = (1 + 2\rho\cos\varphi, \ 2 + \sqrt{2}\rho\sin\varphi)$

$$J_T = \begin{vmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{vmatrix} = \begin{vmatrix} 2\cos\varphi & -2\rho\sin\varphi \\ \sqrt{2}\sin\varphi & \sqrt{2}\rho\cos\varphi \end{vmatrix} = 2\sqrt{2}\,\rho$$

$$\Rightarrow \iint_D (x+y)\,dx\,dy = \iint_A (3+2\rho\cos\varphi+\sqrt{2}\rho\sin\varphi)\,\rho\,d\rho\,d\varphi =$$

$$= \iint_A 3\rho\,d\rho\,d\varphi + \iint_A \rho^2(2\cos\varphi+\sqrt{2}\sin\varphi)\,d\rho\,d\varphi$$

$$= 2\pi\, 3\frac{\rho^2}{2}\Big|_0^1 + \int_0^1 \rho^2\,d\rho \cdot \int_0^{2\pi} \underbrace{(2\cos\varphi+\sqrt{2}\sin\varphi)\,d\varphi}_{0} = 3\pi.$$