ΟΜΑΔΑ Β

Θ.1.: Θέτουμε $u = y - x$, $v = y + x \Leftrightarrow$

$$\boxed{x = \frac{v-u}{2}, \quad y = \frac{v+u}{2}}$$

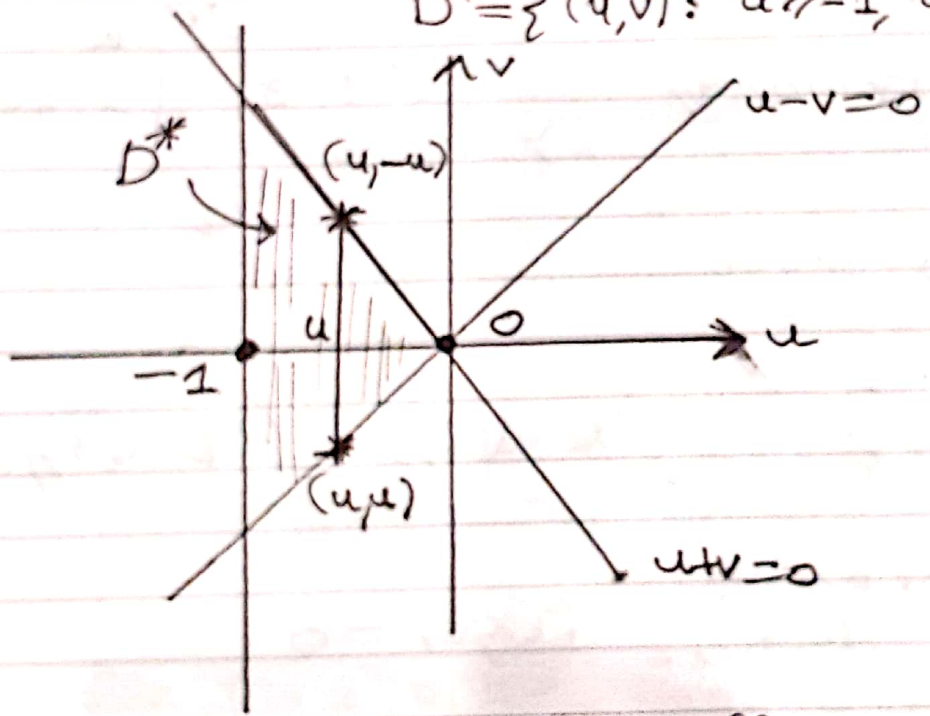
Για $(x,y) \in D$, έχουμε $x \geq 0 \Leftrightarrow \boxed{u - v \leq 0}$,

$y \leq 0 \Leftrightarrow \boxed{u + v \leq 0}$

$y - x \geq -1 \Leftrightarrow \boxed{u \geq -1}$

$\Rightarrow$ το D μετασχηματίζεται στο χωρίο

$$D^* = \{(u,v) : u \geq -1, \ u - v \leq 0, \ u + v \leq 0\}.$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -1/2.$$

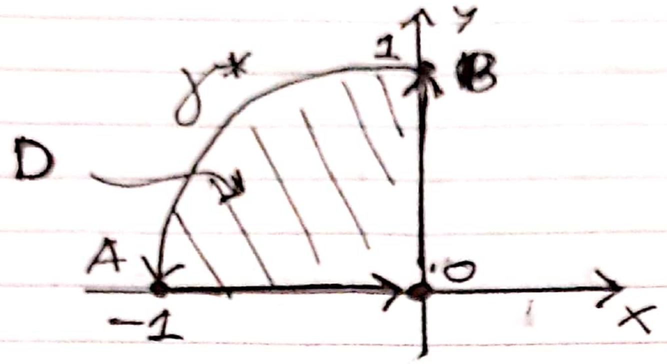
Το ολοκλήρωμα γράφεται $\iint_{D^*} \cos\left(\frac{v}{u}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv =$

$$= \frac{1}{2} \iint_{D^*} \cos\left(\frac{v}{u}\right) du\,dv = \frac{1}{2} \int_{-1}^{0} \left[\int_{u}^{-u} \cos\left(\frac{v}{u}\right) dv \right] du$$

$$= \frac{1}{2} \int_{-1}^{0} u \sin\left(\frac{v}{u}\right) \Big|_{v=u}^{v=-u} du = -\sin 1 \int_{-1}^{0} u\,du =$$

$$= \boxed{\frac{1}{2} \sin 1}$$

Θ.2:



Θεωρούμε την θετικά προσανατολ. κλειστή τμ-λεία καμπύλη

$\Gamma = \gamma + \vec{AO} + \vec{OB}$.

Εάν $P = -y^3 + ye^x$, $Q = \cos y + x^3 + e^x$, από Θ-Green παίρνουμε

$$\int_\Gamma (Pdx + Qdy) = \iint_D (Q_x - P_y)\,dxdy =$$

$$= 3\iint_D (x^2+y^2)\,dxdy \overset{\substack{x=r\cos\varphi\\ y=r\sin\varphi}}{=\!=\!=} 3\int_{\pi/2}^{\pi}\int_0^1 r^3\,dr\,d\varphi$$

$$= \boxed{3\pi/8}, \text{ οπότε}$$

$$\int_\gamma (Pdx+Qdy) = \frac{3\pi}{8} - \int_{\vec{AO}} (Pdx+Qdy) - \int_{\vec{OB}} (Pdx+Qdy).$$

- Για $(x,y) \in \vec{AO}$, έχουμε $y=0$, $dy=0$, $P(x,0)=0$

$$\Rightarrow \int_{\vec{AO}} (Pdx+Qdy) = 0.$$

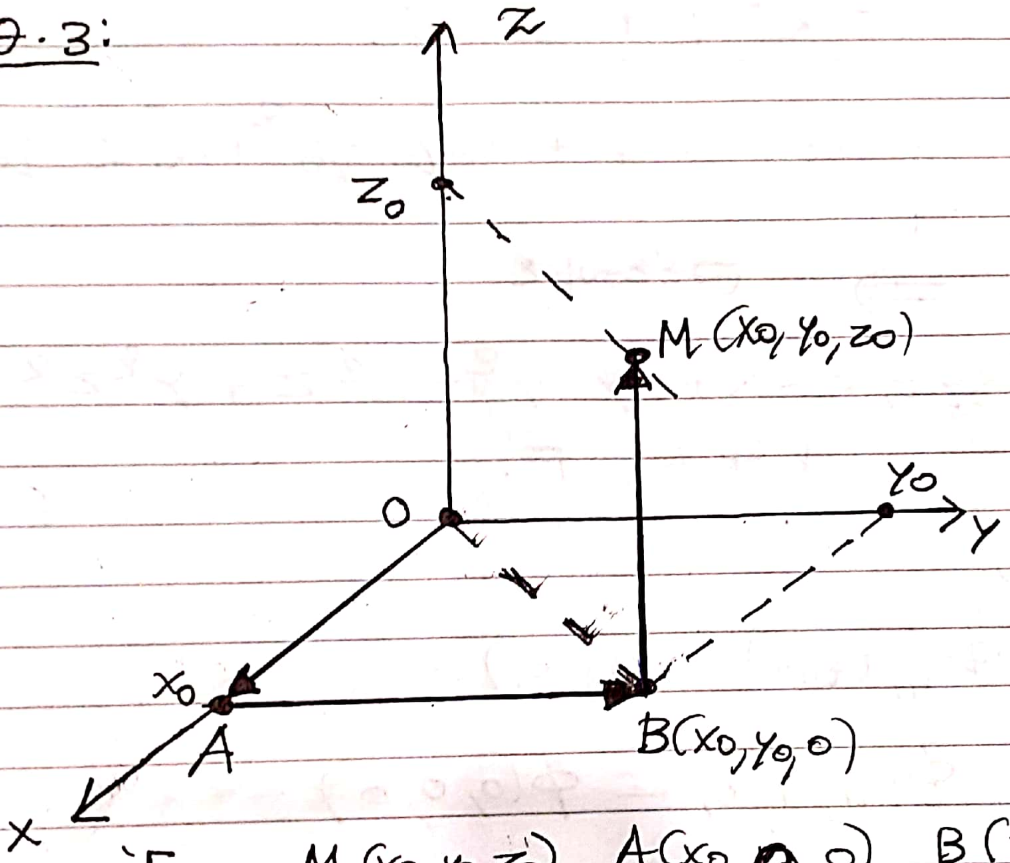
- Για $(x,y) \in \vec{OB}$, έχουμε $x=0$, $dx=0$, $Q(0,y) = \cos y + 1$

$$\Rightarrow \int_{\vec{OB}} (Pdx+Qdy) = \int_0^1 (1+\cos y)\,dy = 1 + \sin 1.$$

Άρα, $\int_\gamma (P\,dx + Q\,dy) = \frac{3\pi}{8} - 0 - 1 - \sin 1$

$$= \boxed{\frac{3\pi}{8} - 1 - \sin 1}.$$

Θ.3:



Έστω $M(x_0, y_0, z_0)$, $A(x_0, 0, 0)$, $B(x_0, y_0, 0) \in \mathbb{R}^3$.
Θεωρούμε την καμπύλη

$$\Gamma = \vec{OA} + \vec{AB} + \vec{BM}.$$

• Για $(x,y,z) \in \vec{OA}$, έχουμε $y = z = 0$, $x \in [0, x_0]$

$$\Rightarrow \int_{\vec{OA}} \vec{F} = \int_0^{x_0} (-1)\,dx = -x_0.$$

• Για $(x,y,z) \in \vec{AB}$, έχουμε $x = x_0$, $y \in [0, y_0]$, $z = 0$

$$\Rightarrow \int_{\vec{AB}} \vec{F} = \int_0^{y_0} (x_0 - 2y)\,dy = x_0 y_0 - y_0^2.$$

• Για $(x,y,z) \in \overrightarrow{BM}$, έχουμε
$x = x_0,\ y = y_0,\ z \in [0, z_0]$

$$\Rightarrow \int_{\overrightarrow{BM}} \vec{F} = \int_0^{z_0} (x_0^2 + 2y_0^2 z)\, dz$$

$$= x_0^2 z_0 + y_0^2 z_0^2.$$

Επομένως, $\int_\Gamma \vec{F} = -x_0 + x_0 y_0 - y_0^2 + x_0^2 z_0 + y_0^2 z_0^2$

$\Longrightarrow$ ~~θ~~έτουμε

$$\Phi(x,y,z) = -x + xy - y^2 + x^2 z + y^2 z^2$$

κ' $\nabla\Phi = \vec{F}$.

Επιπλέον,

$$\int_\gamma \vec{F} = \Phi(\gamma(4\pi)) - \Phi(\gamma(0)) =$$

$$= \Phi(\pi, \pi, 0) - \Phi(0,0,0)$$

$$= -\pi + \pi^2 - \pi^2 = -\pi$$

$\blacksquare$

Θ.4: $\operatorname{div}\vec{F} = \frac{1}{3}\frac{3}{2}(x^2+y^2)^{3/2-1}2x + 0 + 0$

$= x\sqrt{x^2+y^2}.$

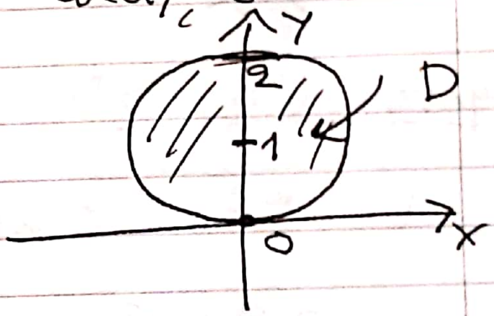
Η ζητούμενη ροή ισούται (λόγω Θ.-Gauss) με

$$\iiint_K \operatorname{div}\vec{F} = \iiint_K x\sqrt{x^2+y^2}\,dx\,dy\,dz =$$

$$= \iint_D \left(\int_0^{x^2+y^2} dz\right) x\sqrt{x^2+y^2}\,dx\,dy$$

$$= \iint_D x\,(x^2+y^2)^{3/2}\,dx\,dy, \text{ όπου}$$

$D = \{(x,y) : x^2+y^2 \le 2y\}.$



Θέτουμε $x = r\cos\varphi$, $y = r\sin\varphi$

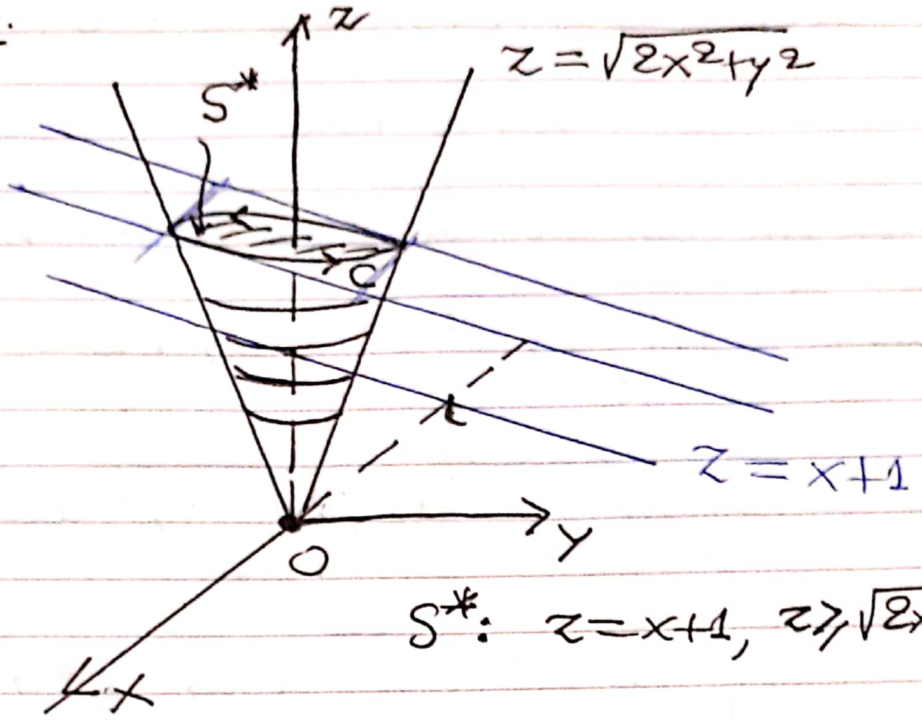
$(x,y)\in D \Rightarrow$ $\underline{0\le\varphi\le\pi}$, $r^2 \le 2r\sin\varphi \Rightarrow \underline{0\le r\le 2\sin\varphi}$

κ' $J = r \Rightarrow$ $\iiint_K \operatorname{div}\vec{F} =$

$$= \int_0^{\pi}\left(\int_0^{2\sin\varphi} r^5\cos\varphi\right) dr\,d\varphi$$

$$= \int_0^{\pi} \frac{2^6}{6}\sin^6\varphi\cos\varphi\,d\varphi = \frac{2^6}{6}\left[\frac{\sin^7\varphi}{7}\right]_0^{\pi} = \boxed{0}.$$

Θ.5.



S^*: $z=x+1$, $z \geq \sqrt{2x^2+y^2}$

Παραμέτρηση της S^*: $S(u,v)=(u,v,1+u)$

με $\sqrt{2u^2+v^2} \leq u+1 \Leftrightarrow 2u^2+v^2 \leq (u+1)^2$

$\Leftrightarrow 2u^2+v^2 \leq u^2+2u+1$

$\Leftrightarrow u^2+v^2-2u \leq 1 \Leftrightarrow (u-1)^2+v^2 \leq 2$,

οπότε

$S:\Delta \to \mathbb{R}^3$, $\Delta=\{(u,v): (u-1)^2+v^2 \leq 2\}$, με

$S(u,v)=(u,v,1+u)$.

Θ. Stokes $\Rightarrow \int_C \vec{F} = \iint_\Delta \operatorname{rot}\vec{F}(S(u,v))\cdot \vec{n}(u,v)\,du\,dv$.

Έχουμε $S_u \times S_v = (-1,0,1)$

κ' $(S_u \times S_v)\cdot S(u,v) = 1 > 0 \Rightarrow$ το $S_u \times S_v$

"δείχνει" προς τα έξω, αν σχεδιασθεί με αρχή

το $S(u,v) \Rightarrow$ $\vec{n}(u,v)=(-1,0,1)$.

Άρα, $\forall (u,v) \in \Delta$,

$\text{rot}\vec{F}(S(u,v)) \cdot \vec{n}(u,v) =$

$= \left(-u^2,\ (u+1)^4 e^{v^2+(u+1)^2},\ v^2\right) \cdot (-1, 0, 1)$

$= u^2+v^2 \quad \Rightarrow \quad \int_C \vec{F} = \iint_\Delta (u^2+v^2)\,du\,dv$

$$\Rightarrow \int_C \vec{F} \underset{v=r\sin\varphi}{\overset{u=r\cos\varphi+1}{=\!=}} \int_0^{2\pi} \int_0^{\sqrt{2}} \left[(1+r\cos\varphi)^2 + r^2\sin^2\varphi\right] r\,dr\,d\varphi$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \left(r^2+1+2r\cos\varphi\right) r\,dr\,d\varphi$$

$$= 2\pi \int_0^{\sqrt{2}} (r^3+r)\,dr + \cancel{2\underbrace{\int_0^{2\pi} \cos\varphi\,d\varphi}_{0} \cdot \int_0^{\sqrt{2}} r^2\,dr}$$

$$= 2\pi \left(\frac{r^4}{4} + \frac{r^2}{2}\right)\Big|_0^{\sqrt{2}} = \boxed{4\pi.}$$