ΟΜΑΔΑ Α

Θ.1: Θέτουμε

$$u = y - x, \quad v = y + x$$

$$\Rightarrow \begin{cases} y = \frac{u+v}{2} \\ x = \frac{v-u}{2}. \end{cases}$$

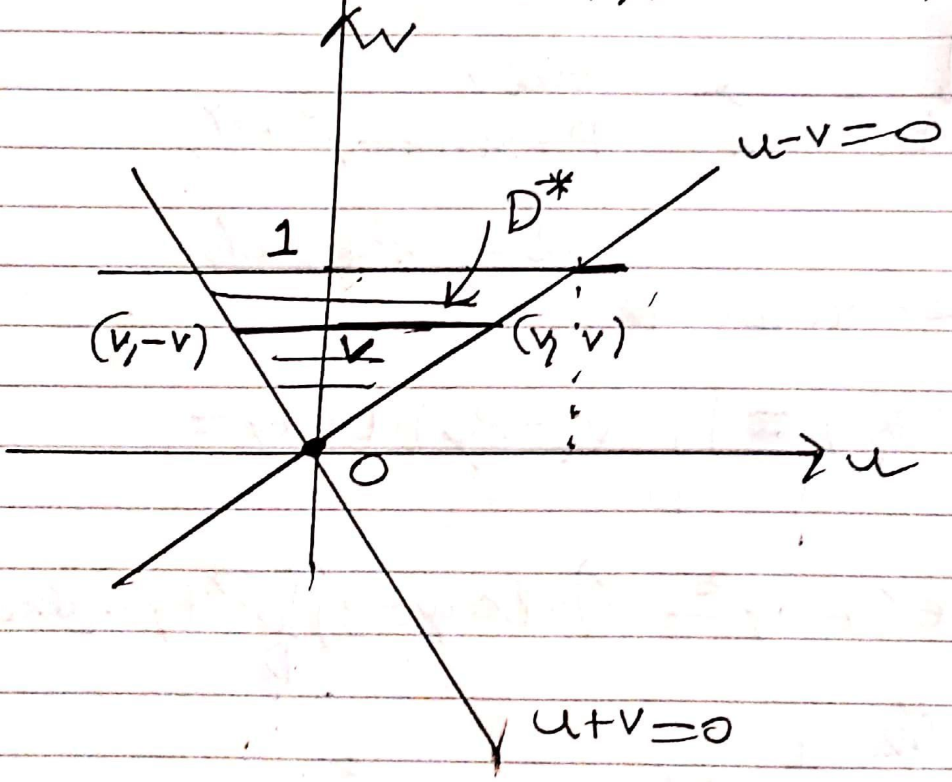
Για $(x,y) \in D$, έχουμε $x \geq 0 \Rightarrow \underline{u - v \leq 0}$

κ/ $x + y \leq 1 \Rightarrow \underline{v \leq 1}$

κ/ $y \geq 0 \Rightarrow \underline{u + v \geq 0}$

$\Rightarrow$ το D μετασχηματίζεται στο χωρίο

$$D^* = \{(u,v) : u - v \leq 0,\ u + v \geq 0,\ v \leq 1\}.$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -1/2 \Rightarrow$$

$$\Rightarrow \iint_D \sin\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_{D^*} \sin\left(\frac{u}{v}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

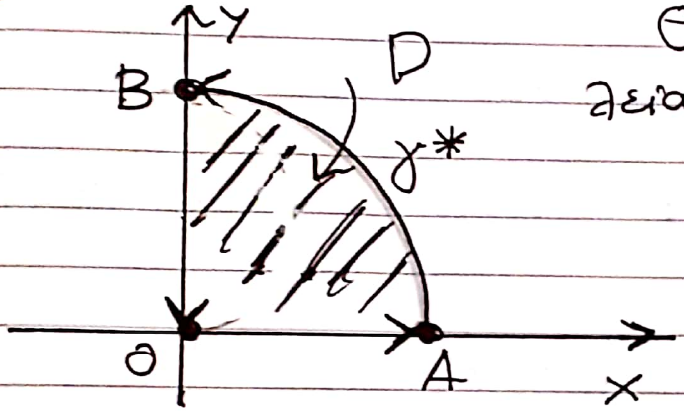
$$= \frac{1}{2} \iint_{D^*} \sin\left(\frac{u}{v}\right) du\,dv = \frac{1}{2} \int_0^1 \left(\int_{-v}^{v} \sin\left(\frac{u}{v}\right) du \right) dv$$

$$= \frac{1}{2}\int_0^1 -v\cos\left(\frac{u}{v}\right)\Big|_{u=-v}^{v} dv$$

$$= \frac{1}{2}\int_0^1 v\left[\cos 1-\cos(-1)\right] dv$$

$$= -\left(\int_0^1 v\,dv\right)\cos 1 = \boxed{-\frac{1}{2}\cos 1}.$$

Θέμα 2:



Θεωρούμε την κλειστή τμ. λεία καμπύλη (θετική προσαν.)

$$\Gamma = \gamma + \vec{BO} + \vec{OA}.$$

Εάν $P = \sin x - y^3 + e^y$,

$Q = x^3 + xe^y$,

από Θ. Green παίρνουμε

$$\oint_\Gamma (P\,dx + Q\,dy) = \iint_D (Q_x - P_y)\,dx\,dy =$$

$$= \iint_D (3x^2 + e^y + 3y^2 - e^y)\,dx\,dy = 3\iint_D (x^2+y^2)\,dx\,dy$$

$$\underset{y=r\sin\varphi}{\overset{x=r\cos\varphi}{=}} 3\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\varphi = \frac{3\pi}{8}$$

$$\Rightarrow \int_\gamma (P\,dx + Q\,dy) = \frac{3\pi}{8} - \int_{\vec{BO}} (P\,dx + Q\,dy) - \int_{\vec{OA}} (P\,dx + Q\,dy)$$

$$= \frac{3\pi}{8} + \int_{\overrightarrow{OB}} (Pdx + Qdy) - \int_{\overrightarrow{OA}} (Pdx + Qdy).$$

Αλλά, για $(x,y) \in \overrightarrow{OB}$, έχουμε $x=0$, $y \in [0,1]$

κ' $Q(x,y) = Q(0,y) = 0$

$$\Rightarrow \int_{\overrightarrow{OB}} (Pdx + Qdy) = 0$$

κ' για $(x,y) \in \overrightarrow{OA}$, έχουμε $y=0$, $x \in [0,1]$

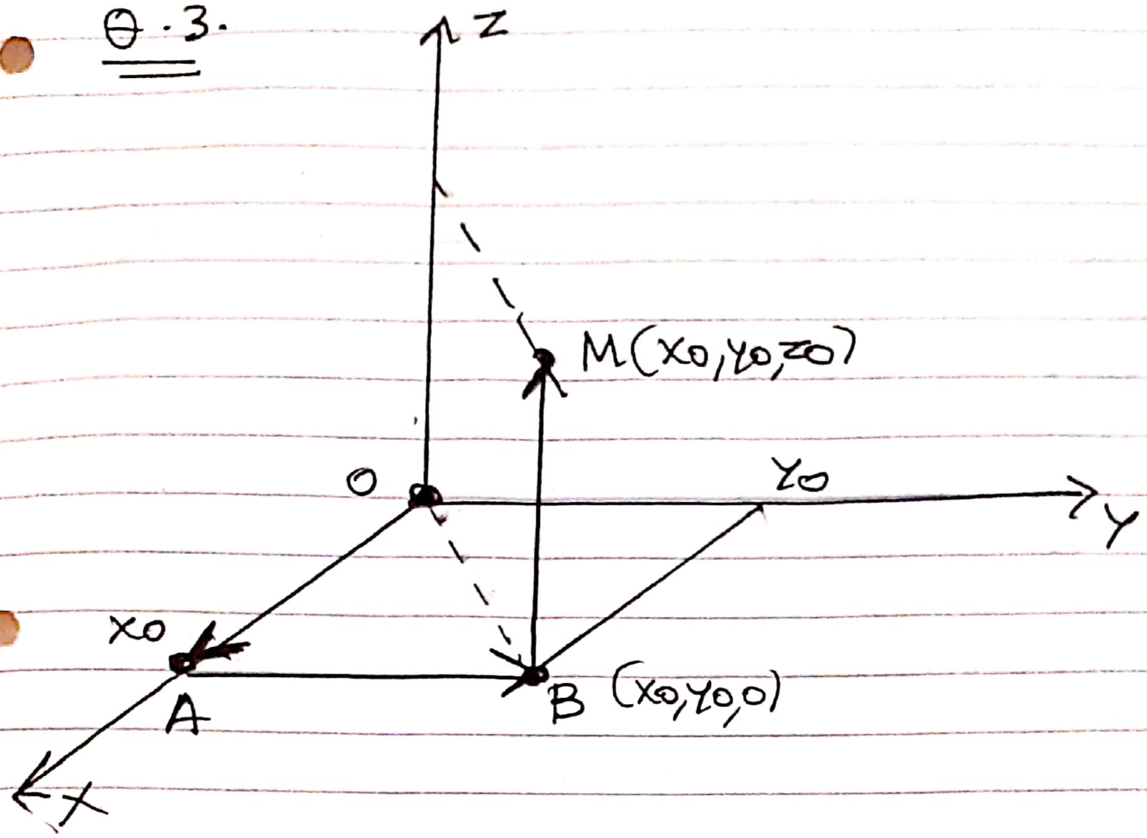
$$\Rightarrow P(x,y) = P(x,0) = \sin x + 1$$

$$\Rightarrow \int_{\overrightarrow{OA}} (Pdx + Qdy) = \int_0^1 (\sin x + 1)\, dx$$

$$= 1 - \cos 1 + 1 = 2 - \cos 1.$$

Άρα, $\int_{\gamma} (Pdx + Qdy) = \boxed{\frac{3\pi}{8} - 2 + \cos 1.}$

<u>Θ.3.</u>



Έστω $M(x_0, y_0, z_0) \in \mathbb{R}^3$, $A(x_0, 0, 0)$, $B(x_0, y_0, 0) \in \mathbb{R}^3$.

Θεωρούμε την καμπύλη $\Gamma = \vec{OA} + \vec{AB} + \vec{BM}$.

- Για $(x,y,z) \in \vec{OA}$, έχουμε $y = z = 0$, $x \in [0, x_0]$

$$\Rightarrow \vec{F}(x,y,z) = \vec{F}(x,0,0) = (-1, \ x^2, \ x) \text{ κ' } dy = dz = 0$$

$$\Rightarrow \int_{\vec{OA}} \vec{F} = \int_0^{x_0} (-1)\, dx = \underline{\underline{-x_0}}.$$

- Για $(x,y,z) \in \vec{AB}$, έχουμε $x = x_0$, $y \in [0, y_0]$, $z = 0$

$$\Rightarrow dx = dz = 0 \text{ κ' } \vec{F}(x,y,z) = \vec{F}(x_0, y, 0) =$$

$$= (2x_0 y - 1, \ x_0^2, \ x_0) \Rightarrow$$

$$\Rightarrow \int_{\vec{AB}} \vec{F} = \int_0^{y_0} x_0^2\,dy = \underline{\underline{x_0^2 y_0}}.$$

• Για $(x,y,z) \in \vec{BM}$, έχουμε $x = x_0$, $y = y_0$, $z \in [0, z_0]$

$$\Rightarrow \underline{dx = dy = 0} \text{ κ' } \vec{F}(x_0, y_0, z) =$$

$$= (2x_0y_0 + z - 1,\ x_0^2 + 2y_0 z^2,\ 2y_0^2 z + x_0 - 2z)$$

$$\Rightarrow \int_{\vec{BM}} \vec{F} = \int_0^{z_0} 2z(y_0^2 - 1)\,dz + x_0 z_0$$

$$= \underline{z_0^2(y_0^2 - 1) + x_0 z_0}$$

Άρα, $\displaystyle\int_{\Gamma} \vec{F} = -x_0 + x_0^2 y_0 + z_0^2(y_0^2 - 1) + x_0 z_0$

$\Rightarrow$ μια συνάρτηση δυναμικού για την $\vec{F}$ είναι η

$$\boxed{\Phi(x,y,z) = -x + x^2 y + z^2(y^2 - 1) + xz}$$

$$\Rightarrow \int_{\gamma} \vec{F} = \Phi(\gamma(4\pi)) - \Phi(\gamma(0))$$

$$= \Phi(\pi, -\pi, 0) - \Phi(0,0,0)$$

$$= -\pi + \pi^2(-\pi) = \boxed{-\pi - \pi^3.}$$

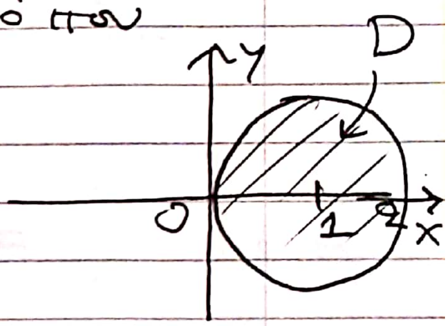
Θ-4. $\operatorname{div}\vec{F} = 0 + \frac{1}{3}\frac{3}{2}(x^2+y^2)^{3/2-1}\, 2y + 0 = y\sqrt{x^2+y^2}.$

Η ζητούμενη ροή ισούται (λόγω Θ-Gauss) με

$$\iiint_K \operatorname{div}\vec{F} = \iiint_K y\sqrt{x^2+y^2}\, dx\,dy\,dz$$

$$= \iint_D \left(\int_0^{x^2+y^2} dz\right) y\sqrt{x^2+y^2}\, dx\,dy$$

$$= \iint_D y\,(x^2+y^2)^{3/2}\, dx\,dy \text{, όπου}$$



$D = \{(x,y) : x^2+y^2 \le 2x\}.$

Θέτουμε $x = r\cos\varphi$, $y = r\sin\varphi$

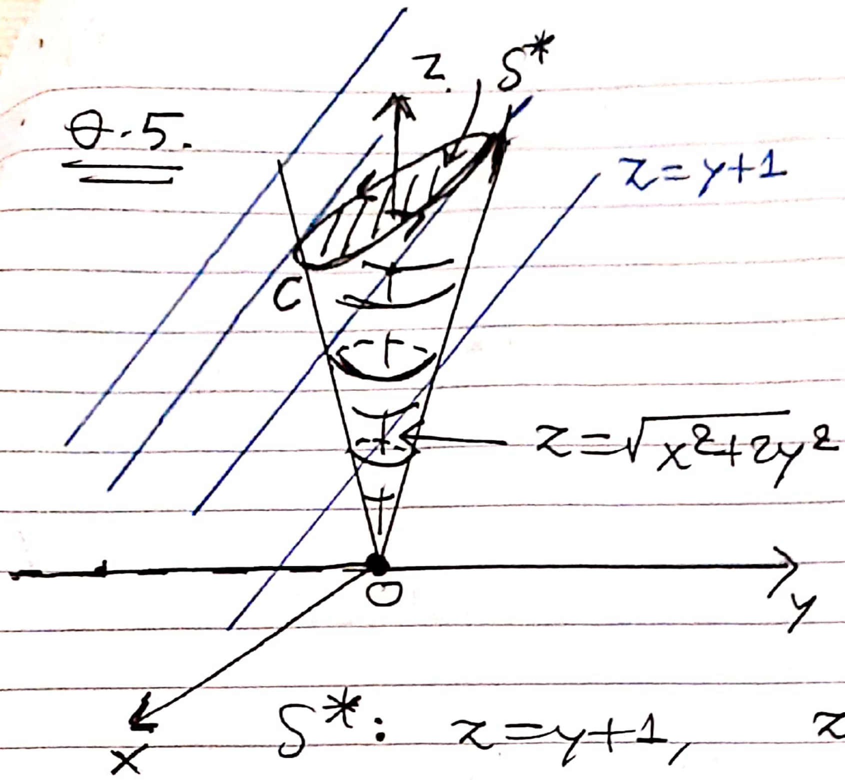
$$(x,y)\in D \Rightarrow -\pi/2 \le \varphi \le \pi/2, \quad r^2 \le 2r\cos\varphi \Rightarrow 0 \le r \le 2\cos\varphi$$

κ/ $J = r \Rightarrow \iiint_K \operatorname{div}\vec{F} =$

$$= \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\cos\varphi} r^5 \sin\varphi\, dr\right) d\varphi$$

$$= \int_{-\pi/2}^{\pi/2} \underbrace{\frac{2^6 \cos^6\varphi}{6}\sin\varphi}_{\text{περιττή}}\, d\varphi = 0.$$

Θ.5.



$S^*: \ z = y+1, \quad z \geq \sqrt{x^2+2y^2}$

Παραμέτρηση της S^*: $S(u,v) = (u, v, v+1)$

με $\sqrt{u^2+2v^2} \leq v+1 \iff u^2+2v^2 \leq (v+1)^2$

$\iff u^2+2v^2 \leq v^2+2v+1$

$\iff u^2+v^2-2v \leq 1$

$\iff u^2+(v-1)^2 \leq 2,$

οπότε $S: \Delta \to \mathbb{R}^3, \ \Delta = \{(u,v): u^2+(v-1)^2 \leq 2\}$

με $S(u,v) = (u, v, v+1)$.

Θ-Stokes $\Rightarrow \int_C \vec{F} = \iint_\Delta \operatorname{rot}\vec{F}(S(u,v)) \cdot \vec{n}(u,v)\,du\,dv$

Έχουμε

$$S_u \times S_v = (0, -1, 1)$$

$\Rightarrow (S_u \times S_v) \cdot S(u,v) = -v+v+1 = 1 > 0$

$\Rightarrow S_u \times S_v$ "δείχνει" προς τα έξω αν σχεδιασθεί με αρχή το $S(u,v) \Rightarrow \vec{n}(u,v) = (0,-1,1)$.

Άρα, $\forall (u,v) \in \Delta$,

$$\text{rot}\vec{F}(S(u,v)) \cdot \vec{n}(u,v) =$$

$$= \left(3(v+1)^2 e^{(v+1)^3+v^2},\ u^2,\ -v^2\right) \cdot (0,-1,1)$$

$$= -(u^2+v^2)$$

$$\Rightarrow \int_C \vec{F} = -\iint_\Delta (u^2+v^2)\,du\,dv =$$

$$\underset{v=1+r\sin\varphi}{\overset{u=r\cos\varphi}{=}} -\int_0^{2\pi}\int_0^{\sqrt{2}} \left[r^2\cos^2\varphi + (1+r\sin\varphi)^2\right] r\,dr\,d\varphi$$

$$= -\int_0^{2\pi}\int_0^{\sqrt{2}} (r^2+2r\sin\varphi+1)\,r\,dr\,d\varphi$$

$$= -2\pi\int_0^{\sqrt{2}} (r^3+r)\,dr + 2\underbrace{\int_0^{2\pi}\sin\varphi\,d\varphi}_{0} \cdot \int_0^{\sqrt{2}} r^2\,dr$$

$$= -2\pi\left(\frac{r^4}{4}+\frac{r^2}{2}\right)\Bigg|_0^{\sqrt{2}} = -2\pi \cdot 2$$

$$= \boxed{-4\pi}.$$