Σύντομες λύσεις ασκήσεων Φυλλαδίου 7

1) (α): $\displaystyle\int \frac{x^3}{x^8+2}\,dx \underset{du=4x^3dx}{\overset{u=x^4}{=}} \frac{1}{4}\int\frac{du}{u^2+2} = \frac{1}{4}\frac{1}{\sqrt{2}}\,\text{Arctan}\left(\frac{u}{\sqrt{2}}\right)+C = \frac{1}{4\sqrt{2}}\,\text{Arctan}\left(\frac{x^4}{\sqrt{2}}\right)+C,\ x\in\mathbb{R}.$

(β): $\displaystyle\int\frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx \underset{du=\frac{dx}{2\sqrt{x}}}{\overset{u=\sqrt{x}}{=}} 2\int\cos(u)\,du = 2\sin(u)+C = 2\sin(\sqrt{x})+C,\ x>0.$

(γ): $\displaystyle\int\frac{\ln(\ln x)}{x}\,dx \underset{du=\frac{dx}{x}}{\overset{u=\ln(x)}{=}} \int\ln(u)\,du = \int u'\ln(u)\,du = u\ln(u)-\int u\cdot\frac{1}{u}\,du = u\ln(u)-u+C = \ln(x)\ln[\ln(x)]-\ln(x)+C,$
$(x>1)$

(δ): $\displaystyle\int\frac{dx}{2x^2+3x+2} = \int\frac{dx}{2\left(x^2+\frac{3x}{2}+1\right)} = \frac{1}{2}\int\frac{dx}{x^2+2\cdot\frac{3x}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+1} = \frac{1}{2}\int\frac{dx}{\left(x+\frac{3}{4}\right)^2+\frac{7}{16}} =$

$\displaystyle = \frac{1}{2}\,\frac{16}{7}\int\frac{dx}{\left(\frac{x+3/4}{\sqrt{7/16}}\right)^2+1} \overset{\frac{x+3/4}{\sqrt{7/16}}=u}{=} \frac{8}{7}\int\frac{(\sqrt{7/16})\,du}{u^2+1} = \frac{1}{2\sqrt{7}}\int\frac{du}{u^2+1} = \frac{1}{2\sqrt{7}}\,\text{Arctan}(u)+C =$

$\displaystyle = \frac{1}{2\sqrt{7}}\,\text{Arctan}\left(\frac{16x+12}{\sqrt{7}}\right)+C,\quad x\in\mathbb{R}.$

(ε): $\int\sqrt{e^x-1}\,dx$. Θέτουμε $u=\sqrt{e^x-1}$. Πρέπει $e^x\geq 1$, άρα, $x\geq 0$. Έχουμε $u^2+1=e^x \Rightarrow 2u\,du=e^x\,dx \Rightarrow$
$\Rightarrow dx=\frac{2u}{u^2+1}\,du$. Άρα, $\displaystyle\int\sqrt{e^x-1}\,dx = \int u\cdot\frac{2u}{u^2+1}\,du = 2\int\frac{u^2}{u^2+1}\,du = 2\int\frac{u^2+1-1}{u^2+1}\,du = 2\int\left(1-\frac{1}{u^2+1}\right)du =$
$= 2\left(u-\text{Arctan}(u)\right)+C = 2\sqrt{e^x-1}-2\,\text{Arctan}\left(\sqrt{e^x-1}\right)+C,\quad x\geq 1.$

(ζ): $\displaystyle\int x^4[\ln(x)]^2\,dx = \int\left(\frac{x^5}{5}\right)'[\ln(x)]^2\,dx = \frac{x^5}{5}[\ln(x)]^2-\int\frac{x^5}{5}\cdot\frac{2\ln(x)}{x}\,dx = \frac{x^5}{5}[\ln(x)]^2-\frac{2}{5}\int x^4\ln(x)\,dx =$
$\displaystyle = \frac{x^5}{5}[\ln(x)]^2-\frac{2}{5}\int\left(\frac{x^5}{5}\right)'\ln(x)\,dx = \frac{x^5}{5}[\ln(x)]^2-\frac{2}{5}\left[\frac{x^5}{5}\ln(x)-\int\frac{x^5}{5}\cdot\frac{1}{x}\,dx\right] =$
$\displaystyle = \frac{x^5}{5}[\ln(x)]^2-\frac{2}{25}x^5\ln(x)+\frac{2}{25}\int x^4\,dx = \frac{x^5}{5}[\ln(x)]^2-\frac{2}{25}x^5\ln(x)+\frac{2}{125}x^5+C,\quad x>0.$

(η) $\displaystyle\int xe^{2x}\sin(3x)\,dx = \int\left(\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}\right)'\sin(3x)\,dx$, αφού $\displaystyle\int xe^{2x}\,dx = \int x\cdot\left(\frac{1}{2}e^{2x}\right)'dx =$
$\displaystyle = \frac{x}{2}e^{2x}-\int\frac{1}{2}e^{2x}\,dx = \frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}+C.$ Άρα, $\displaystyle\int xe^{2x}\sin(3x)\,dx$ = (παραγοντική)
$\displaystyle = \left(\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}\right)\sin(3x)-\int\left(\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}\right)3\cos(3x)\,dx = \left(\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}\right)\sin(3x)\,-$
$\displaystyle -3\int\left(\frac{x}{4}e^{2x}-\frac{1}{8}e^{2x}-\frac{1}{8}e^{2x}\right)'\cos(3x)\,dx = \left(\frac{x}{2}e^{2x}-\frac{1}{4}e^{2x}\right)\sin(3x)-3\left(\frac{x}{4}e^{2x}-\frac{1}{4}e^{2x}\right)\cos(3x)$
$\displaystyle -9\int\left(\frac{x}{4}e^{2x}-\frac{1}{4}e^{2x}\right)\sin(3x)\,dx = -\frac{9}{4}\int xe^{2x}\sin(3x)\,dx+\frac{9}{4}\int\sin(3x)e^{2x}\,dx+F(x)\quad (1)$

όπου $F(x)=\left(\frac{x}{2}-\frac{1}{4}\right)e^{2x}\sin(3x)-\frac{3}{4}(x-1)e^{2x}\cos(3x). \Rightarrow \left(1+\frac{9}{4}\right)\int e^{2x}x\sin(3x)dx=$

$=\frac{9}{4}\int e^{2x}\sin(3x)dx+F(x) \Rightarrow \int xe^{2x}\sin(3x)dx=\frac{9}{13}\int e^{2x}\sin(3x)dx+\frac{4}{13}F(x)$

Υπολογίζουμε τώρα το $\int e^{2x}\sin(3x)dx=\int\left(\frac{1}{2}e^{2x}\right)'\sin(3x)dx=\frac{1}{2}e^{2x}\sin(3x)-\int\frac{1}{2}e^{2x}\cdot 3\cos(3x)dx=$

$=\frac{1}{2}e^{2x}\sin(3x)-\frac{3}{2}\int\left(\frac{1}{2}e^{2x}\right)'\cos(3x)dx=\frac{1}{2}e^{2x}\sin(3x)-\frac{3}{2}\left[\frac{1}{2}e^{2x}\cos(3x)+\int\frac{1}{2}e^{2x}\sin(3x)\cdot 3\,dx\right]=$

$=\frac{1}{2}e^{2x}\sin(3x)-\frac{3}{4}e^{2x}\cos(3x)-\frac{9}{4}\int e^{2x}\sin(3x)dx \Rightarrow \left(1+\frac{9}{4}\right)\int e^{2x}\sin(3x)dx=$

$=\frac{1}{2}e^{2x}\sin(3x)-\frac{3}{4}e^{2x}\cos(3x)+c \Rightarrow \int e^{2x}\sin(3x)dx=\frac{2}{13}e^{2x}\sin(3x)-\frac{3}{13}e^{2x}\cos(3x)+c,\ x\in\mathbb{R}$

Τελικά, $\int xe^{2x}\sin(3x)dx=\frac{18}{169}e^{2x}\sin(3x)-\frac{27}{169}e^{2x}\cos(3x)+\frac{4}{13}\left(\frac{x}{2}-\frac{1}{4}\right)e^{2x}\sin(3x)-\frac{3}{13}(x-1)e^{2x}\cos(3x)+c$

$(x\in\mathbb{R})$

(θ): $\int\sin[\ln(x)]dx=\int x'\sin[\ln(x)]dx=x\sin[\ln(x)]-\int x\cdot\cos[\ln(x)]\cdot\frac{1}{x}dx=x\sin[\ln(x)]-\int\cos[\ln(x)]dx=$

$=x\sin[\ln(x)]-\int x'\cos[\ln(x)]dx=x\sin[\ln(x)]-\left[x\cos[\ln(x)]+\int x\sin[\ln(x)]\cdot\frac{1}{x}dx\right]=$

$=x\sin[\ln(x)]-x\cos[\ln(x)]-\int\sin[\ln(x)]dx \Rightarrow 2\int\sin[\ln(x)]dx=x\sin[\ln(x)]-x\cos[\ln(x)]+c$

$\Rightarrow \int\sin[\ln(x)]dx=\frac{x}{2}\sin[\ln(x)]-\frac{x}{2}\cos[\ln(x)]+c\ ,\ x>0$

(ι): $\int[\mathrm{Arcsin}(x)]^2dx=\int x'[\mathrm{Arcsin}(x)]^2dx=x[\mathrm{Arcsin}(x)]^2-\int x\cdot\frac{2\,\mathrm{Arcsin}(x)}{\sqrt{1-x^2}}dx=$

$=x[\mathrm{Arcsin}(x)]^2+\int\left(2\sqrt{1-x^2}\right)'\mathrm{Arcsin}(x)dx=x[\mathrm{Arcsin}(x)]^2+2\sqrt{1-x^2}\,\mathrm{Arcsin}(x)-\int2\sqrt{1-x^2}\,\frac{1}{\sqrt{1-x^2}}dx=$

$=x[\mathrm{Arcsin}(x)]^2+2\sqrt{1-x^2}\,\mathrm{Arcsin}(x)-2x+c\ ,\ -1<x<1$

(κ): $\int\frac{dx}{(x^2+1)^3}=\int\frac{1+x^2-x^2}{(x^2+1)^3}dx=\int\frac{dx}{(1+x^2)^2}-\int\frac{x^2}{(1+x^2)^3}dx=\int\frac{dx}{(1+x^2)^2}-\int x\cdot\frac{x}{(1+x^2)^3}dx$

Έχουμε $\int x\cdot\frac{x}{(1+x^2)^3}dx=\int x\cdot\left[\frac{1}{4}\,\frac{1}{(1+x^2)^2}\right]'dx=\frac{x}{4(1+x^2)^2}-\int\frac{dx}{4(1+x^2)^2} \Rightarrow$

$\Rightarrow \int\frac{dx}{(x^2+1)^3}=\int\frac{dx}{(1+x^2)^2}+\frac{x}{4(1+x^2)^2}-\frac{1}{4}\int\frac{dx}{(1+x^2)^2}=\frac{3}{4}\int\frac{dx}{(1+x^2)^2}+\frac{x}{4(1+x^2)^2}=$

$=\frac{3}{4}\left[\frac{1}{2}\mathrm{Arctan}(x)+\frac{x}{2(1+x^2)}\right]+\frac{x}{4(1+x^2)^2}+c=\frac{3}{8}\mathrm{Arctan}(x)+\frac{3x}{8(1+x^2)}+\frac{x}{4(1+x^2)^2}+c$

$(x\in\mathbb{R})$

Άσκηση 22

2) (α): $\int \frac{x^5+2}{x^2-1}\,dx$:

$$\begin{array}{r|l} x^5+2 & x^2-1 \\ \cline{2-2} -x^5+x^3 & x^3+x \\ \hline x^3+2 & \\ -x^3+x & \\ \hline x+2 & \end{array}$$

$\Rightarrow x^5+2=(x^3+x)(x^2-1)+x+2 \Rightarrow$

$\Rightarrow \frac{x^5+2}{x^2-1} = x^3+x+\frac{x+2}{x^2-1} = x^3+x+\frac{x+1+1}{x^2-1} =$

$= x^3+x+\frac{1}{x-1}+\frac{1}{x^2-1}$

$\Rightarrow \int \frac{x^5+2}{x^2-1}\,dx = \int \left(x^3+x+\frac{1}{x-1}+\frac{1}{x^2-1}\right)dx = \frac{x^4}{4}+\frac{x^2}{2}+\ln|x-1|+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C,\ x\in I$

I διάστημα που δεν περιέχει τα $-1, 1$.

(β): $\int \frac{dx}{x^3-4x^2}$: $\frac{1}{x^3-4x^2} = \frac{1}{x^2(x-4)} = \frac{A}{x}+\frac{B}{x^2}+\frac{\Gamma}{x-4},\ \forall x\neq 0,4.$

$B=\lim_{x\to 0}\frac{1}{x-4} = -\frac{1}{4}$. $\Gamma = \lim_{x\to 4}\frac{1}{x^2} = \frac{1}{16}$. Θέτουμε $x=1$ στην αρχική.

Έχουμε: $\frac{1}{1^2(1-4)} = A+B+\frac{\Gamma}{1-4} \Rightarrow -\frac{1}{3} = A+B-\frac{\Gamma}{3} \Rightarrow A+B=\frac{\Gamma-1}{3} \Rightarrow B=-A+\frac{\Gamma-1}{3} \Rightarrow$

$\Rightarrow A = -B+\frac{\Gamma-1}{3} = \frac{1}{4}-\frac{1}{3}+\frac{1}{48} \Rightarrow A = -\frac{1}{16}$. Συνεπώς, $\int\frac{dx}{x^3-4x^2} =$

$= -\frac{1}{16}\int\frac{dx}{x} - \frac{1}{4}\int\frac{dx}{x^2}+\frac{1}{16}\int\frac{dx}{x-4} = -\frac{1}{16}\ln|x|+\frac{1}{4x^2}+\left(\frac{1}{16}\right)\ln|x-4|+C =$

$= \frac{1}{4x^2}+\frac{1}{16}\ln\left|\frac{x-4}{x}\right|+C,\ x\in I$, I διάστημα που δεν περιέχει τα $0, 4$.

(γ): $\int\frac{dx}{x^3+1}$: $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1}+\frac{Bx+\Gamma}{x^2-x+1},\ \forall x\neq -1.$

$A = \lim_{x\to -1}\frac{1}{x^2-x+1} = \frac{1}{3}$. Επίσης, $\frac{x}{x^3+1} = A\frac{x}{x+1}+\frac{Bx^2+\Gamma x}{x^2-x+1} \xRightarrow{x\to+\infty} 0 = A+B \Rightarrow$

$\Rightarrow B=-A=-1/3$. Για $x=0$, η αρχική δίνει: $1 = A+\Gamma \Rightarrow \Gamma = 1-A = 2/3$

Άρα, $\int\frac{dx}{x^3+1} = \frac{1}{3}\int\frac{dx}{x+1}+\int\frac{-\frac{x}{3}+\frac{2}{3}}{x^2-x+1}\,dx = \frac{1}{3}\ln|x+1|+\frac{1}{3}\int\frac{-x+2}{x^2-x+1}\,dx =$

$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\int\frac{2x-4}{x^2-x+1}\,dx = \frac{1}{3}\ln|x-1|-\frac{1}{6}\int\frac{2x-1-3}{x^2-x+1}\,dx = \frac{1}{3}\ln|x-1|-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\,dx +$

$+\frac{1}{2}\int\frac{dx}{x^2-x+1} = \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{2}\int\frac{dx}{(x-\frac{1}{2})^2+3/4} = \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2-x+1)+$

$+\frac{1}{2}\cdot\frac{4}{3}\int\frac{dx}{\left(\frac{x-1/2}{\sqrt{3}/2}\right)^2+1} = \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\,\mathrm{Arctan}\left(\frac{2x-1}{\sqrt{3}}\right)+C,\ x\in I$

I διάστημα που δεν περιέχει το 1

(δ): $\int \frac{2x^2+2x-7}{(x-2)(x^2+1)^2}$. $\frac{2x^2+2x-7}{(x-2)(x^2+1)^2} = \frac{A}{x-2} + \frac{Bx+\Gamma}{x^2+1} + \frac{\Delta x+E}{(x^2+1)^2}$, $\forall x \neq 2$

$A = \lim_{x\to 2} \frac{2x^2+2x-7}{(x^2+1)^2} = \frac{5}{25} = \frac{1}{5}$. Ακόμα έχουμε: $\frac{2x^2+2x-7}{x-2} = \frac{A(x^2+1)^2}{x-2} + (Bx+\Gamma)(x^2+1) +$

$+ \Delta x + E$. Θέτουμε $x=i \Rightarrow \frac{2i^2+2i-7}{i-2} = \Delta i + E \Rightarrow 2i-9 = (\Delta i+E)(i-2)$

$\Rightarrow 2i-9 = \Delta i^2 - 2\Delta i + Ei - 2E = (E-2\Delta)i - \Delta - 2E \Rightarrow \left.\begin{array}{l} E-2\Delta=2 \\ -\Delta-2E=-9 \end{array}\right\} \Rightarrow$

$\Rightarrow \left.\begin{array}{l} E-2\Delta=2 \\ \Delta+2E=9 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2E-4\Delta=4 \\ \Delta+2E=9 \end{array}\right\} \Rightarrow \begin{array}{l} \Delta=1 \\ E=4 \end{array}$

Πολλ/ντας με x τα δύο μέλη της αρχικής: $\frac{(2x^2+2x-7)x}{(x-2)(x^2+1)^2} = \frac{Ax}{x-2} + \frac{Bx^2+\Gamma x}{x^2+1} + \frac{(\Delta x+E)x}{(x^2+1)^2}$

$\xRightarrow{x\to+\infty} 0 = A+B \Rightarrow B = -A = -1/5$.

Θέτουμε $x=0$ στην αρχική: $\frac{-7}{-2} = -\frac{A}{2} + \Gamma + E \Rightarrow \Gamma = \frac{A+7}{2} - E = -2/5$

Επομένως, $\int \frac{2x^2+2x-7}{(x-2)(x^2+1)^2}dx = \frac{1}{5}\int\frac{dx}{x-2} + \int \frac{-\frac{x}{5}-\frac{2}{5}}{x^2+1}dx + \int\frac{x+4}{(x^2+1)^2}dx =$

$= \frac{1}{5}\ln|x-2| - \frac{1}{10}\int\frac{2x}{x^2+1}dx - \frac{2}{5}\int\frac{dx}{x^2+1} + \frac{1}{2}\int\frac{2x}{(x^2+1)^2}dx + 4\int\frac{dx}{(x^2+1)^2} =$

$= \frac{1}{5}\ln|x-2| - \frac{1}{10}\ln(x^2+1) - \frac{2}{5}\mathrm{Arctan}(x) - \frac{1}{2(1+x^2)} + 4\underbrace{\left[\frac{1}{2}\mathrm{Arctan}(x) + \frac{x}{2(1+x^2)}\right]}_{\text{Άσκηση 22}} + c$

$= \frac{1}{5}\ln|x-2| - \frac{1}{10}\ln(x^2+1) + \frac{8}{5}\mathrm{Arctan}(x) + \frac{4x-1}{2(1+x^2)} + c$, $x\in I$

I διάστημα που δεν περιέχει το 2.

(ε): $\int\frac{dx}{x^4+1}$. $\frac{1}{x^4+1} = \frac{1}{x^4+1+2x^2-2x^2} = \frac{1}{(x^2+1)^2-2x^2} = \frac{1}{(x^2+1-x\sqrt{2})(x^2+1+x\sqrt{2})} =$

$= \frac{A_1x+B_1}{x^2-x\sqrt{2}+1} + \frac{A_2x+B_2}{x^2+x\sqrt{2}+1}$, αφού τα τριώνυμα $x^2-x\sqrt{2}+1$ και $x^2+x\sqrt{2}+1$

έχουν διακρίνουσα $2-4=-2<0$. Επίσης, $x^2-x\sqrt{2}+1=0 \Leftrightarrow x = \frac{\sqrt{2}\pm\sqrt{-2}}{2} = \frac{\sqrt{2}\pm i\sqrt{2}}{2}$

και $x^2+x\sqrt{2}+1=0 \Leftrightarrow x = \frac{-\sqrt{2}\pm i\sqrt{2}}{2}$. Άρα οι ρίζες του $x^2-x\sqrt{2}+1$ είναι οι $\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$

και του $x^2+x\sqrt{2}+1$ είναι οι $-\frac{1}{\sqrt{2}} \pm i\cdot\frac{1}{\sqrt{2}}$.

Θέτουμε $\rho = \frac{1}{\sqrt2} + \frac{1}{\sqrt2} i$. Τότε, $\bar\rho = \frac{1}{\sqrt2} - \frac{1}{\sqrt2} i$ και οι $\rho, \bar\rho$ είναι οι ρίζες του $x^2 - x\sqrt2 + 1$, ενώ οι $-\rho, -\bar\rho$ είναι οι ρίζες του $x^2 + x\sqrt2 + 1$. Συνεπώς, $x^4+1 = (x-\rho)(x-\bar\rho)(x+\rho)(x+\bar\rho)$. Άρα,

$$\frac{1}{(x-\rho)(x-\bar\rho)(x+\rho)(x+\bar\rho)} = \frac{A_1 x + B_1}{(x-\rho)(x-\bar\rho)} + \frac{A_2 x + B_2}{(x+\rho)(x+\bar\rho)}, \quad \forall x \neq \rho, -\rho, \bar\rho, -\bar\rho.$$

$$\Rightarrow \frac{1}{(x-\bar\rho)(x+\rho)(x+\bar\rho)} = \frac{A_1 x + B_1}{x - \bar\rho} + \frac{A_2 x + B_2}{(x+\rho)(x+\bar\rho)}(x-\rho). \quad \text{Για } x \to \rho \Rightarrow$$

$$\Rightarrow \frac{1}{(\rho-\bar\rho)2\rho(\rho+\bar\rho)} = \frac{A_1\rho + B_1}{\rho - \bar\rho} \Rightarrow A_1\rho + B_1 = \frac{1}{2\rho(\rho+\bar\rho)} = \frac{1}{2\rho\cdot\frac{2}{\sqrt2}} = \frac{1}{2\rho\sqrt2}$$

$$\Rightarrow A_1 \rho\, 2\rho\sqrt2 + B_1\, 2\rho\sqrt2 = 1 \Rightarrow A_1 2\sqrt2\,\rho^2 + B_1 2\sqrt2\,\rho = 1 \Rightarrow$$

$$\Rightarrow A_1 2\sqrt2\left(\frac12 - \frac12 + i\right) + B_1 2\sqrt2\left(\frac{1}{\sqrt2} + \frac{1}{\sqrt2} i\right) = 1 \Rightarrow$$

$$\Rightarrow A_1 2\sqrt2\, i + 2B_1 + 2B_1 i = 1 \Rightarrow 2B_1 = 1, \quad 2B_1 + 2A_1\sqrt2 = 0 \Rightarrow$$

$$\Rightarrow B_1 = 1/2, \quad A_1 = -\frac{1}{2\sqrt2}$$

Περαιτέρω, $\frac{1}{(x-\rho)(x-\bar\rho)(x+\bar\rho)} = \frac{A_1 x + B_1}{(x-\rho)(x-\bar\rho)}(x+\rho) + \frac{A_2 x + B_2}{x+\bar\rho}$. Για $x \to -\rho \Rightarrow$

$$\Rightarrow \frac{1}{-2\rho(-\rho-\bar\rho)(\bar\rho-\rho)} = \frac{-A_2\rho + B_2}{\bar\rho - \rho} \Rightarrow -A_2\rho + B_2 = \frac{1}{2\rho(\rho+\bar\rho)}. \quad \text{Άρα,}$$

$-A_2\rho + B_2 = A_1\rho + B_1$. Αναγκαία, αφού A_1, B_1, A_2, B_2 είναι πραγματικοί, ενώ $\rho \notin \mathbb{R}$, πρέπει $-A_2 = A_1$ και $B_2 = B_1 \Rightarrow A_2 = \frac{1}{2\sqrt2}$, $B_2 = 1/2$. Άρα τελικά,

$$\frac{1}{x^4+1} = \frac{-\frac{1}{2\sqrt2}x + \frac12}{x^2 - x\sqrt2 + 1} + \frac{\frac{1}{2\sqrt2}x + \frac12}{x^2 + x\sqrt2 + 1}, \quad \forall x \in \mathbb{R} \Rightarrow$$

$$\Rightarrow \int \frac{dx}{x^4+1} = -\frac{1}{2\sqrt2}\int \frac{x - \sqrt2}{x^2 - x\sqrt2 + 1}\,dx + \frac{1}{2\sqrt2}\int \frac{x+\sqrt2}{x^2 + x\sqrt2 + 1}\,dx =$$

$$= -\frac{1}{4\sqrt2}\int \frac{2x - 2\sqrt2}{x^2 - x\sqrt2 + 1}\,dx + \frac{1}{4\sqrt2}\int \frac{2x + 2\sqrt2}{x^2 + x\sqrt2 + 1}\,dx =$$

$$= -\frac{1}{4\sqrt2}\int \frac{2x-\sqrt2}{x^2-x\sqrt2+1}dx + \frac14\int\frac{dx}{x^2-x\sqrt2+1} + \frac{1}{4\sqrt2}\int\frac{2x+\sqrt2}{x^2+x\sqrt2+1}dx + \frac14\int\frac{dx}{x^2+x\sqrt2+1} =$$

$$= -\frac{1}{4\sqrt2}\ln(x^2-x\sqrt2+1) + \frac{1}{4\sqrt2}\ln(x^2+x\sqrt2+1) + \frac14\int\frac{dx}{x^2-x\sqrt2+1} + \frac14\int\frac{dx}{x^2+x\sqrt2+1}$$

$$= \frac{1}{4\sqrt2}\ln\left(\frac{x^2+x\sqrt2+1}{x^2-x\sqrt2+1}\right) + \frac14\int\frac{dx}{\left(x-\frac{1}{\sqrt2}\right)^2+\frac12} + \frac14\int\frac{dx}{\left(x+\frac{1}{\sqrt2}\right)^2+\frac12} =$$

$$= \frac{1}{4\sqrt2}\ln\left(\frac{x^2+x\sqrt2+1}{x^2-x\sqrt2+1}\right) + \frac{1}{2\sqrt2}\operatorname{Arctan}(x\sqrt2-1) + \frac{1}{2\sqrt2}\operatorname{Arctan}(x\sqrt2+1) + C$$

$(x\in\mathbb{R})$

(ζ): $\displaystyle\int\frac{1-\sqrt{3x+4}}{1+\sqrt{3x+4}}dx \;\underset{\substack{u^2=3x+4\\ 2u\,du=3\,dx}}{\overset{u=\sqrt{3x+4}}{=}}\; \int\frac{1-u}{1+u}\,\frac23 u\,du = \frac23\int\frac{u-u^2}{u+1}du.$

$$u-u^2 \,|\, u+1 \;=\; -u^2+u \,|\, u+1$$

$$\begin{array}{r|l} -u^2+u & u+1 \\ \underline{u^2+u} & -u+2 \\ 2u & \\ \underline{-2u-2} & \\ -2 & \end{array}$$

$\Rightarrow u-u^2 = (u+1)(-u+2)-2 \Rightarrow$

$\Rightarrow \dfrac{u-u^2}{u+1} = 2-u-\dfrac{2}{u+1} \Rightarrow$

$$\Rightarrow \frac23\int\frac{u-u^2}{u+1}du = \frac23\int\left(2-u-\frac{2}{u+1}\right)du = \frac23\left[2u-\frac12u^2-2\ln|1+u|\right]+C =$$

$$= \frac43\sqrt{3x+4} - \frac13(3x+4) - \frac43\ln(1+\sqrt{3x+4}) + C \quad , \quad x\geq -4/3.$$

(η): $\displaystyle\int\frac{e^{2x}+e^x-1}{e^{2x}-2e^x-3}dx \;\underset{dx=\frac{du}{u}}{\overset{e^x=u}{=}}\; \int\frac{u^2+u-1}{u^2-2u-3}\,\frac1u\,du.$

$$\frac{u^2+u-1}{u(u^2-2u-3)} = \frac{u^2+u-1}{u(u+1)(u-3)} = \frac{A}{u}+\frac{B}{u+1}+\frac{\Gamma}{u-3}, \quad \forall u\neq 0,-1,3.$$

$$\Rightarrow A = \lim_{u\to0}\frac{u^2+u-1}{u^2-2u-3} = 1/3, \; B = \lim_{u\to-1}\frac{u^2+u-1}{u(u-3)} = \frac{-1}{(-1)(-4)} = -1/4, \; \Gamma = \lim_{u\to3}\frac{u^2+u-1}{u(u+1)} = \frac{11}{12}$$

$$\Rightarrow \int\frac{u^2+u-1}{u^2-2u-3}\,\frac1u\,du = \frac13\int\frac{du}{u} - \frac14\int\frac{du}{u+1} + \frac{11}{12}\int\frac{du}{u-3} = \frac13\ln|u| - \frac14\ln|u+1| + \frac{11}{12}\ln|u-3| + C$$

$= \frac13\ln(e^x) - \frac14\ln(e^x+1) + \frac{11}{12}\ln|e^x-3| + C$, $x\in I$, I διάστημα που δεν περιέχει το $\ln(3)$. Άρα, $\displaystyle\int\frac{e^{2x}+e^x-1}{e^{2x}-2e^x-3}dx = \frac{x}{3} - \frac14\ln(e^x+1) + \frac{11}{12}\ln|e^x-3| + C$

$\forall x\in I$, $\ln(3)\notin I$.

(θ): $\int \frac{e^x}{(e^x-1)^2+4e^x}dx \underset{e^x dx = du}{\overset{e^x=u}{=}} \int \frac{du}{(u-1)^2+4u} = \int \frac{du}{u^2+2u+1} = \int \frac{du}{(u+1)^2} = -\frac{1}{u+1}+c =$

$= -\frac{1}{e^x+1}+c, \quad x\in\mathbb{R}.$

3) (a): $\int \frac{dx}{\sin(x)} \underset{du=-\sin x dx}{\overset{u=\cos(x)}{=}} -\int \frac{-\sin(x)}{\sin^2(x)}dx = -\int \frac{du}{1-u^2} = \int \frac{du}{u^2-1} = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+c =$

$= \frac{1}{2}\ln\left(\frac{1-\cos(x)}{1+\cos(x)}\right)+c, \quad x\in I$, I διάστημα που δεν περιέχει ρίζες του sin.

(b): $\int \frac{dx}{\sin^3(x)} \underset{du=-\sin x dx}{\overset{u=\cos(x)}{=}} -\int \frac{-\sin x\, dx}{\sin^4(x)} = -\int \frac{du}{(u^2-1)^2}$

$\frac{1}{(u^2-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{\Gamma}{u+1} + \frac{\Delta}{(u+1)^2}, \quad \forall u \neq 1,-1$

$B = \lim_{u\to 1} \frac{1}{(u+1)^2} = \frac{1}{4}. \quad \Delta = \lim_{u\to -1} \frac{1}{(u-1)^2} = \frac{1}{4}.$

$\frac{1}{(u-1)(u+1)^2} = A + \frac{B}{u-1} + \frac{\Gamma(u-1)}{u+1} + \frac{\Delta(u-1)}{(u+1)^2} \overset{u\to+\infty}{\Longrightarrow} 0 = A+\Gamma$

Για $u=0$ η αρχική δίνει: $1 = -A+B+\Gamma+\Delta \Rightarrow 1 = \Gamma - A + \frac{1}{2} \Rightarrow \Gamma - A = \frac{1}{2}$ $\Bigg\} \Rightarrow \Gamma = \frac{1}{4}, \ A = -\frac{1}{4}$

$\Rightarrow -\int \frac{du}{(u^2-1)^2} = \frac{1}{4}\int \frac{du}{u-1} - \frac{1}{4}\int \frac{du}{(u-1)^2} - \frac{1}{4}\int \frac{du}{u+1} - \frac{1}{4}\int \frac{du}{(u+1)^2} =$

$= \frac{1}{4}\ln|u-1| + \frac{1}{4(u-1)} - \frac{1}{4}\ln|u+1| + \frac{1}{4(u+1)} + c = \frac{1}{4}\ln\left(\frac{1-\cos(x)}{1+\cos(x)}\right) + \frac{1}{4}\frac{1}{\cos(x)-1} + \frac{1}{4}\frac{1}{\cos(x)+1} + c$

$x\in I$, I διάστημα χωρίς ρίζες του sin.

(γ): $\int \sin^4(x)\cos^5(x)dx \underset{du=\cos(x)dx}{\overset{u=\sin(x)}{=}} \int \sin^4(x)\cos^4(x)\cos(x)dx = \int u^4(1-u^2)^2 du = \int u^4(1+u^4-2u^2)du =$

$= \int (u^4+u^8-2u^6)du = \frac{u^5}{5} + \frac{u^9}{9} - \frac{2u^7}{7} + c = \frac{1}{5}\sin^5(x) + \frac{1}{9}\sin^9(x) - \frac{2}{7}\sin^7(x) + c, \quad x\in\mathbb{R}.$

(δ): $\int \sin^4(x)dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \int \frac{1+\cos^2(2x)-2\cos(2x)}{4}dx = \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{1}{4}\int \cos^2(2x)dx$

$= \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{1}{4}\int \frac{1+\cos(4x)}{2}dx = \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{x}{8} + \frac{1}{32}\sin(4x) + c =$

$= \frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c, \quad x\in\mathbb{R}.$

(ε): $\int \frac{\sin^2(x)}{1+\cos^2(x)}dx$. Θέτουμε $t=\tan(x) \Rightarrow dt = \frac{dx}{\cos^2(x)} = (t^2+1)dx \Rightarrow$
$\Rightarrow dx = \frac{dt}{1+t^2}$. Επίσης, $\int \frac{\sin^2 x}{1+\cos^2 x}dx =$

$$= \int \frac{\sin^2(x)/\cos^2(x)}{\frac{1}{\cos^2(x)}+1}dx = \int \frac{t^2}{1+1+t^2}\frac{dt}{1+t^2} = \int \frac{t^2}{(1+t^2)(2+t^2)}dt =$$

$$= \int \frac{t^2+1-1}{(1+t^2)(2+t^2)}dt = \int\left[\frac{1}{2+t^2} - \frac{1}{(1+t^2)(2+t^2)}\right]dt = \int \frac{dt}{2+t^2} - \int \frac{dt}{(1+t^2)(2+t^2)} =$$

$$= \frac{1}{\sqrt{2}}\operatorname{Arctan}\left(\frac{t}{\sqrt{2}}\right) - \int\left(\frac{1}{1+t^2} - \frac{1}{2+t^2}\right)dt = \sqrt{2}\operatorname{Arctan}\left(\frac{t}{\sqrt{2}}\right) - \operatorname{Arctan}(t) + C$$

$= \sqrt{2}\operatorname{Arctan}\left(\frac{\tan(x)}{\sqrt{2}}\right) - x + C$, $x \in I$, I διάστημα χωρίς ρίζες του cos.

(ϛ): $$\int \frac{dx}{2\sin^2(x)+3\cos^2(x)} = \int \frac{dx}{2+\cos^2(x)} \overset{t=\tan(x)}{\underset{dx=\frac{dt}{1+t^2}}{=}} \int \frac{\frac{dt}{1+t^2}}{2+\frac{1}{1+t^2}} = \int \frac{dt}{2t^2+3} =$$

$$= \frac{1}{2}\int \frac{dt}{t^2+3/2} = \frac{1}{2}\frac{\sqrt{2}}{\sqrt{3}}\operatorname{Arctan}\left(\frac{t\sqrt{2}}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{6}}\operatorname{Arctan}\left(\frac{\sqrt{2}}{\sqrt{3}}\tan(x)\right) + C,$$

$x \in I$ διάστημα χωρίς ρίζες του cos.

(η): $\int \frac{dx}{2+\sin(x)}$, $-\pi<x<\pi$. Θέτουμε $t=\tan(x/2) \Rightarrow dx = \frac{2dt}{1+t^2}$, $\sin(x) = \frac{2t}{1+t^2}$

Άρα, $$\int \frac{dx}{2+\sin(x)} = \int \frac{\frac{2dt}{1+t^2}}{2+\frac{2t}{1+t^2}} = \int \frac{2dt}{2+2t^2+2t} = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$= \frac{4}{3}\int \frac{dt}{\left(\frac{t+1/2}{\sqrt{3}/2}\right)^2+1} = \frac{2}{\sqrt{3}}\operatorname{Arctan}\left(\frac{2t+1}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\operatorname{Arctan}\left(\frac{2\tan(x/2)+1}{\sqrt{3}}\right) + C$$

(θ): $\int \frac{1+\cos(x)}{2+\sin(x)}dx$, $-\pi<x<\pi$, $t=\tan(x/2)$, $\sin(x) = \frac{2t}{1+t^2}$, $\cos(x) = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

$$\Rightarrow \int \frac{1+\cos(x)}{2+\sin(x)}dx = \int \frac{1+\frac{1-t^2}{1+t^2}}{2+\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt = \int \frac{2}{2+2t+2t^2}\frac{2}{1+t^2}dt = 2\int \frac{dt}{(1+t^2)(1+t+t^2)}$$

$$\frac{1}{(1+t^2)(1+t+t^2)} = \frac{At+B}{1+t^2} + \frac{\Gamma t+\Delta}{1+t+t^2}, \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \frac{1}{1+t+t^2} = At+B + \frac{\Gamma t+\Delta}{1+t+t^2}(t^2+1).$$ Θέτουμε $t=i$, έχουμε:

$$\frac{1}{1+i+i^2} = Ai+B \Rightarrow Ai+B = \frac{1}{i} \Rightarrow -A+Bi=1 \Rightarrow A=-1,\ B=0$$

Επίσης, $$\frac{t}{(1+t^2)(1+t+t^2)} = \frac{At^2+Bt}{1+t^2} + \frac{\Gamma t^2+\Delta t}{1+t+t^2} \xrightarrow{t\to+\infty} 0 = A+\Gamma \Rightarrow \Gamma=-A=1$$

Για $t=0$ η αρχική γίνεται: $1 = B+\Delta \Rightarrow \Delta = 1-B = 1$

Άρα, $$\int \frac{dt}{(1+t^2)(1+t+t^2)} = -\int \frac{t}{1+t^2}dt + \int \frac{t+1}{1+t^2+t}dt = -\frac{1}{2}\int \frac{2t}{1+t^2}dt + \frac{1}{2}\int \frac{2t+2}{1+t^2+t}dt$$

$$= -\frac{1}{2}\ln(1+t^2) + \frac{1}{2}\int \frac{2t+1}{1+t+t^2}dt + \int \frac{dt}{1+t^2+t} = -\frac{1}{2}\ln(1+t^2) + \frac{1}{2}\ln(1+t+t^2) +$$

$$+\frac{2}{\sqrt{3}}\text{Arctan}\left(\frac{2t+1}{\sqrt{3}}\right) + C$$ (όπως στην (4)). Συνεπώς,

$$\int \frac{1+\cos(x)}{2+\sin(x)}dx = -\ln\left[1+\tan^2(x/2)\right] + \ln\left[1+\tan(x/2)+\tan^2(x/2)\right] + \frac{4}{\sqrt{3}}\text{Arctan}\left(\frac{2\tan(x/2)+1}{\sqrt{3}}\right) + C$$

$-\pi < x < \pi$.

(7): $\int \frac{1-\sin(x)}{1+\sin(x)}dx$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Θέτουμε $t=\tan(x/2) \Rightarrow dx = \frac{2dt}{1+t^2}$ και

$\sin(x) = \frac{2t}{1+t^2}$. Άρα, $$\int \frac{1-\sin x}{1+\sin x}dx = \int \frac{1-\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt = \int \frac{1+t^2-2t}{1+t^2+2t}\frac{2}{1+t^2}dt$$

$$\frac{2(1+t^2-2t)}{(1+t^2+2t)(1+t^2)} = \frac{2(1+t^2-2t)}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{\Gamma t+\Delta}{1+t^2},\ \forall t\neq -1.$$

$$B = \lim_{t\to -1}\frac{2(1+t^2-2t)}{1+t^2} = 4.$$ Επίσης, $$\frac{2(1+t^2-2t)}{1+t^2+2t} = \frac{A(1+t^2)}{1+t} + \frac{B(1+t^2)}{(1+t)^2} + \Gamma t+\Delta$$

$$\overset{t=i}{\Rightarrow} \frac{2(1+i^2-2i)}{1+i^2+2i} = \Gamma i+\Delta \Rightarrow \frac{-4i}{2i} = \Gamma i+\Delta \Rightarrow \Gamma i+\Delta = -2 \Rightarrow \Gamma=0,\ \Delta=-2.$$

Για $t=0$ η αρχική δίνει: $2 = A+B+\Delta \Rightarrow A = 2-B-\Delta = 0 \Rightarrow$

$$\int \frac{1+t^2-2t}{1+t^2+2t}\cdot\frac{2}{1+t^2}dt = 4\int\frac{dt}{(1+t)^2} + \int\frac{-2}{1+t^2}dt = -\frac{4}{1+t} - 2\text{Arctan}(t) + C =$$

$$= \frac{-4}{1+\tan(x/2)} - x + C,\quad -\pi/2 < x < \pi/2$$

4) (a): $\int \frac{dx}{\sqrt{x^2-3x+2}}$, $x<1$: $x^2-3x+2=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$. Άρα, $\int \frac{dx}{\sqrt{x^2-3x+2}} =$

$$= \int \frac{dx}{\sqrt{(x-3/2)^2-1/4}} = \int \frac{dx}{\frac{1}{2}\sqrt{\left(\frac{x-3/2}{1/2}\right)^2-1}} = 2\int \frac{dx}{\sqrt{\left(\frac{x-3/2}{1/2}\right)^2-1}} \overset{\frac{x-3/2}{1/2}=u}{\underset{dx=\frac{1}{2}du}{=}} \int \frac{du}{\sqrt{u^2-1}}$$

Αφού $x<1 \Rightarrow \frac{x-3/2}{1/2}<-1$. Δηλαδή, $u<-1$. Θέτουμε $u=-\cosh(t)$, $t>0$

$\Rightarrow \sqrt{u^2-1}=\sqrt{\cosh^2(t)-1}=\sinh(t)$, αφού $t>0$. Άρα, $\int \frac{du}{\sqrt{u^2-1}} = \int \frac{-\sinh(t)dt}{\sinh(t)} =$

$$= -t+c = -\ln\left(-u+\sqrt{u^2-1}\right)+c = -\ln\left[3-2x+\sqrt{(2x-3)^2-1}\right]+c, \quad x<1$$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2-3x+2}} = -\ln\left[3-2x+2\sqrt{x^2-3x+2}\right]+c, \quad x<1.$$

(b): $\int \frac{dx}{x\sqrt{x^2-x+1}}$, $x>0$: $\int \frac{dx}{x\sqrt{x^2-x+1}} = \int \frac{dx}{x\sqrt{(x-1/2)^2+3/4}} = \int \frac{dx}{x\frac{\sqrt{3}}{2}\sqrt{\left(\frac{x-1/2}{\sqrt{3}/2}\right)^2+1}}$

Θέτουμε $u=\frac{x-1/2}{\sqrt{3}/2}$. Αφού $x>0$, πρέπει $u>-1/\sqrt{3}$. Άρα,

$$\int \frac{dx}{x\sqrt{x^2-x+1}} = \int \frac{(\sqrt{3}/2)\,du}{\left(\frac{u\sqrt{3}}{2}+\frac{1}{2}\right)\frac{\sqrt{3}}{2}\sqrt{u^2+1}} = 2\int \frac{du}{(u\sqrt{3}+1)\sqrt{u^2+1}} \underset{\substack{du=\cosh(t)dt\\ \sqrt{u^2+1}=\cosh(t)}}{\overset{u=\sinh(t)}{=}}$$

$$= 2\int \frac{\cosh(t)\,dt}{\cosh(t)\left[\sqrt{3}\sinh(t)+1\right]} = 2\int \frac{dt}{\sqrt{3}\,\frac{e^t-e^{-t}}{2}+1} = 2\int \frac{dt}{\frac{\sqrt{3}}{2}(e^{2t}-1)\frac{1}{e^t}+1} =$$

$$= 2\int \frac{e^t\,dt}{\frac{\sqrt{3}}{2}e^{2t}-\frac{\sqrt{3}}{2}+e^t} \underset{e^t dt=dv}{\overset{e^t=v}{=}} 2\int \frac{dv}{\frac{\sqrt{3}}{2}v^2+v-\frac{\sqrt{3}}{2}} = 4\int \frac{dv}{\sqrt{3}v^2+2v-\sqrt{3}} =$$

$$= 4\int \frac{dv}{\sqrt{3}\left(v-\frac{1}{\sqrt{3}}\right)(v+\sqrt{3})} = \frac{4}{\sqrt{3}}\int \left(\frac{1}{v-1/\sqrt{3}}-\frac{1}{v+\sqrt{3}}\right)\frac{\sqrt{3}}{4}\,dv = \int \frac{dv}{v-1/\sqrt{3}} - \int \frac{dv}{v+\sqrt{3}} =$$

$= \ln\left|\frac{v-1/\sqrt{3}}{v+\sqrt{3}}\right|+c = \ln\left(\frac{e^t-1/\sqrt{3}}{e^t+\sqrt{3}}\right)+c$, αφού $\sinh(t)=u>-\frac{1}{\sqrt{3}}$, $v=e^t$

$\Rightarrow v-\frac{1}{v}>\frac{-2}{\sqrt{3}} \Rightarrow \sqrt{3}v^2+2v-\sqrt{3}>0 \Rightarrow v>\frac{1}{\sqrt{3}}$ αφού $v>0$.

$$\Rightarrow \int \frac{dx}{x\sqrt{x^2-x+1}} = \ln\left(\frac{u+\sqrt{u^2+1}-1/\sqrt{3}}{u+\sqrt{u^2+1}+\sqrt{3}}\right)+c = \ln\left[\frac{\frac{2x-1}{\sqrt{3}}+\sqrt{\frac{(2x-1)^2}{3}+1}-1/\sqrt{3}}{\frac{2x-1}{\sqrt{3}}+\sqrt{\frac{(2x-1)^2}{3}+1}+\sqrt{3}}\right]+c$$

με $x>0$

(γ): $\int \frac{\sqrt{x^2+6x+5}}{x}dx$, $x>0$. Έχουμε ότι $\sqrt{x^2+6x+5} = \sqrt{(x+3)^2-4} = 2\sqrt{\left(\frac{x+3}{2}\right)^2-1}$

Θέτουμε $u = \frac{x+3}{2}$. Αφού $x>0 \Rightarrow u>3/2$. Άρα, $\int \frac{\sqrt{x^2+6x+5}}{x} = 2\int \frac{\sqrt{u^2-1}}{2u-3}\cdot 2du =$

$$= 4\int \frac{\sqrt{u^2-1}}{2u-3}du \underset{t>0}{\overset{u=\cosh(t)}{=}} 4\int \frac{\sinh(t)\cdot\sinh(t)dt}{2\cosh(t)-3} = 4\int \frac{(e^t-e^{-t})^2\cdot\frac{1}{4}}{2\frac{e^t+e^{-t}}{2}-3}dt = \int \frac{(e^t-e^{-t})^2}{e^t+e^{-t}-3}dt =$$

$$= \int \frac{(e^{2t}-1)^2 e^{-2t}}{e^{2t}-3e^t+1}e^t\,dt \overset{e^t=v}{=} \int \frac{(v^2-1)^2}{v^2(v^2-3v+1)}dv = \int \frac{v^4-2v^2+1}{v^2(v^2-3v+1)}dv =$$

$$= \int \left[1+\frac{3v^3-3v^2+1}{v^2(v^2-3v+1)}\right]dv = v + \int \frac{3v^3-3v^2+1}{v^2(v^2-3v+1)}dv.$$

$$\frac{3v^3-3v^2+1}{v^2(v^2-3v+1)} = \frac{A}{v}+\frac{B}{v^2}+\frac{\Gamma}{v-\rho_1}+\frac{\Delta}{v-\rho_2}, \quad \forall v\neq 0, \rho_1, \rho_2, \text{ όπου}$$

ρ_1, ρ_2 οι ρίζες του $v^2-3v+1=0$, δηλαδή, $\rho_1 = \frac{3+\sqrt5}{2}$, $\rho_2 = \frac{3-\sqrt5}{2}$.

$$B = \lim_{v\to 0}\frac{3v^3-3v^2+1}{v^2-3v+1} = 1, \quad \Gamma = \lim_{v\to\rho_1}\frac{3v^3-3v^2+1}{v^2(v-\rho_2)} = \frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2(\rho_1-\rho_2)} = \frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt5}$$

$$\Delta = \lim_{v\to\rho_2}\frac{3v^3-3v^2+1}{v^2(v-\rho_1)} = \frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2(\rho_2-\rho_1)} = -\frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt5}$$

Αφού η αρχική είναι: $\frac{3v^3-3v^2+1}{v(v^2-3v+1)} = A+\frac{B}{v}+\frac{\Gamma v}{v-\rho_1}+\frac{\Delta v}{v-\rho_2} \xrightarrow{v\to+\infty} 3 = A+\Gamma+\Delta$

$$\Rightarrow A = 3-\Gamma-\Delta = 3-\frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt5}+\frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt5} = 3+\frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt5}-\frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt5}$$

$$= 3+\frac{1}{\sqrt5}\frac{3\rho_1^2\rho_2^3-3\rho_1^2\rho_2^2+\rho_1^2-3\rho_1^3\rho_2^2+3\rho_1^2\rho_2^2-\rho_2^2}{\rho_1^2\rho_2^2} = 3+\frac{1}{\sqrt5}\frac{3\rho_1^2\rho_2^2(\rho_2-\rho_1)+\rho_1^2-\rho_2^2}{\rho_1^2\rho_2^2} =$$

$$= 3+\frac{1}{\sqrt5}\frac{(\rho_1-\rho_2)(\rho_1+\rho_2-3\rho_1^2\rho_2^2)}{\rho_1^2\rho_2^2} = 3+\frac{1}{\sqrt5}(\rho_1-\rho_2)\frac{3-3}{\rho_1^2\rho_2^2} = 3 \Rightarrow$$

$$\frac{3v^3-3v^2+1}{v^2(-3v+v^2+1)} = \frac{3}{v}+\frac{1}{v^2}+\frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt5}\frac{1}{v-\rho_1}-\frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt5}\frac{1}{v-\rho_2} \Rightarrow$$

$$\int \frac{3v^3-3v^2+1}{v^2(v^2-3v+1)}dv = 3\ln(v) - \frac{1}{v} + \frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt{5}}\ln(v-\rho_1) - \frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt{5}}\ln(v-\rho_2)+C$$

(Αφού $u > \frac{3}{2} \Rightarrow v^2-3v+1>0$ με $v=e^t>0 \Rightarrow v>\rho_1>\rho_2$). Άρα,

$$\int \frac{\sqrt{x^2+6x+5}}{x}dx = v + 3\ln(v) - \frac{1}{v} + \frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt{5}}\ln(v-\rho_1) - \frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt{5}}\ln(v-\rho_2)+C$$

$$= e^t + 3t - \frac{1}{e^t} + \frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt{5}}\ln(e^t-\rho_1) - \frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt{5}}\ln(e^t-\rho_2)+C$$

$$= u+\sqrt{u^2-1} + 3\ln(u+\sqrt{u^2-1}) - \frac{1}{u+\sqrt{u^2-1}} + \frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt{5}}\ln(u+\sqrt{u^2-1}-\rho_1) -$$

$$- \frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt{5}}\ln(u+\sqrt{u^2-1}-\rho_2)+C =$$

$$= \frac{x+3}{2} + \sqrt{\left(\frac{x+3}{2}\right)^2-1} + 3\ln\left[\frac{x+3}{2}+\sqrt{\left(\frac{x+3}{2}\right)^2-1}\right] - \frac{1}{\frac{x+3}{2}+\sqrt{\left(\frac{x+3}{2}\right)^2-1}} +$$

$$+ \frac{3\rho_1^3-3\rho_1^2+1}{\rho_1^2\sqrt{5}}\ln\left[\frac{x+3}{2}+\sqrt{\left(\frac{x+3}{2}\right)^2-1}-\rho_1\right] - \frac{3\rho_2^3-3\rho_2^2+1}{\rho_2^2\sqrt{5}}\ln\left[\frac{x+3}{2}+\sqrt{\left(\frac{x+3}{2}\right)^2-1}-\rho_2\right]+C$$

για $x>0$.

(δ): $\displaystyle\int \frac{dx}{2x+\sqrt{x^2+4}}$, $x > \frac{-2}{\sqrt{3}}$ (Άρα $2x+\sqrt{x^2+4} > 0$)

$$\int \frac{dx}{2x+\sqrt{x^2+4}} \overset{x=2\sinh(t)}{=} \int \frac{2\cosh(t)\,dt}{4\sinh(t)+2\cosh(t)} = \int \frac{e^t+e^{-t}}{2(e^t-e^{-t})+e^t+e^{-t}}dt = \int \frac{(e^{2t}-1)e^{-t}}{3e^t-e^{-t}}dt$$

$$= \int \frac{e^{2t}-1}{3e^{2t}-1}dt \underset{dt=\frac{du}{u}}{\overset{e^t=u}{=}} \int \frac{u^2-1}{3u^2-1}\frac{du}{u}$$

[Παρατήρηση: $\sinh(t) = \frac{x}{2} > \frac{-1}{\sqrt{3}} \Leftrightarrow e^t > \frac{1}{\sqrt{3}} \Leftrightarrow u > 1/\sqrt{3}$]

$$\frac{u^2-1}{(3u^2-1)u} = \frac{A}{u} + \frac{B}{u-1/\sqrt{3}} + \frac{\Gamma}{u+1/\sqrt{3}}, \quad \forall u \neq 0, 1/\sqrt{3}, -1/\sqrt{3}$$

$$A = \lim_{u\to 0} \frac{u^2-1}{3u^2-1} = 1, \quad B = \lim_{u\to 1/\sqrt{3}} \frac{u^2-1}{u\cdot 3(u+1/\sqrt{3})} = \frac{1/3-1}{3/\sqrt{3}\cdot 2/\sqrt{3}} = -1/3$$

$$B = \lim_{u\to -1/\sqrt{3}} \frac{u^2-1}{u\cdot 3(u-1/\sqrt{3})} = \frac{1/3-1}{-1/\sqrt{3}\cdot 3 \cdot \frac{-2}{\sqrt{3}}} = -1/3 \xRightarrow{u>1/\sqrt{3}}$$

$$\int \frac{u^2-1}{(3u^2-1)u}\,du = + \ln(u) - \frac{1}{3}\ln\left(u-\frac{1}{\sqrt{3}}\right) - \frac{1}{3}\ln\left(u+\frac{1}{\sqrt{3}}\right) + C =$$

$$= t - \frac{1}{3}\ln\left(e^t - \frac{1}{\sqrt{3}}\right) - \frac{1}{3}\ln\left(e^t+\frac{1}{\sqrt{3}}\right) + C = t - \frac{1}{3}\ln\left(e^{2t}-\frac{1}{3}\right) + C$$

$$= \ln\left[\frac{x}{2}+\sqrt{\frac{x^2}{4}+1}\right] - \frac{1}{3}\ln\left(\left(\frac{x}{2}+\sqrt{\frac{x^2}{4}+1}\right)^2 - \frac{1}{3}\right) + C, \quad x > -\frac{2}{\sqrt{3}}.$$

(Є): $\displaystyle\int \sqrt{4-x^2}\,dx \underset{-\frac{\pi}{2}<t<\frac{\pi}{2}}{\overset{x=2\sin(t)}{=}} \int \sqrt{4-4\sin^2 t}\; 2\cos(t)\,dt = \int 2\cos(t)\cdot 2\cos(t)\,dt = 4\int \cos^2(t)\,dt =$

$$= 4\int \frac{1+\cos(2t)}{2}\,dt = 2\int (1+\cos(2t))\,dt = 2t + \sin(2t) + C = 2\,\mathrm{Arcsin}\left(\frac{x}{2}\right) + 2\sin(t)\cos(t) + C$$

$$= 2\,\mathrm{Arcsin}\left(\frac{x}{2}\right) + x\sqrt{1-\frac{x^2}{4}} + C = 2\,\mathrm{Arcsin}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + C, \quad -2<x<2$$

(З): $\displaystyle\int \frac{dx}{\sqrt{2x+3-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(\frac{x-1}{2}\right)^2}} \underset{du=\frac{dx}{2}}{\overset{u=\frac{x-1}{2}}{=}} \int \frac{du}{\sqrt{1-u^2}} =$

$$= \mathrm{Arcsin}(u) + C = \mathrm{Arcsin}\left(\frac{x-1}{2}\right) + C, \quad |x-1|<2.$$