Τριγωνομετρικές Αντικαταστάσεις

Αν $R(x,y) = \dfrac{P(x,y)}{Q(x,y)}$, όπου $P(x,y)$ και $Q(x,y)$ πολυώνυμα δύο μεταβλητών, τότε λέμε ότι η $R(x,y)$ είναι ρητή συνάρτηση των x,y. Έτσι, η συνάρτηση $R(\sin(x), \cos(x))$, όπου $R(x,y)$, λέγεται ρητή συνάρτηση των $\sin(x)$, $\cos(x)$. Για να υπολογίσουμε το ~~$\int R(x,y)$~~ $\int R(\sin(x), \cos(x))dx$ θα το ανάγουμε, με κατάλληλη αντικατάσταση, σε ένα ολοκλήρωμα ρητής συνάρτησης μίας μεταβλητής.

Περίπτωση 1: Η $R(\sin x, \cos x)$ είναι περιττή ως προς $\sin x$: Δηλαδή,

$R(-\sin x, \cos x) = -R(\sin x, \cos x)$. Τότε θέτουμε $u = \cos x$.

Περίπτωση 2: Η $R(\sin x, \cos x)$ είναι περιττή ως προς $\cos x$: $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

Τότε θέτουμε $u = \sin x$

Περίπτωση 3: Η $R(\sin x, \cos x)$ είναι άρτια ως προς $\sin x$ και ως προς $\cos x$. Θέτουμε

$\tan(x) = u$ $\quad \left(R(-\sin x, -\cos x) = R(\sin x, \cos x) \right)$.

Περίπτωση 4: Γενικά. Για το $\int R(\sin x, \cos x)dx$, όταν η $R(\sin x, \cos x)$ δεν εμπίπτει σε κάποια από τις 3 προηγούμενες περιπτώσεις, θέτουμε $t = \tan\left(\frac{x}{2}\right)$, όταν $-\pi < x < \pi$.

Παραδείγματα: 1) $\displaystyle\int \frac{dx}{\cos(x)} \underset{du=\cos(x)dx}{\overset{u=\sin(x)}{=\!=}} \int \frac{\cos(x)}{\cos^2(x)}dx = \int \frac{du}{1-u^2} = -\int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)\frac{1}{2}du =$

$$= -\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + c = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + c,$$

x ανήκει σε διάστημα που δεν περιέχει κανένα από τους $2k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$, και $2k\pi + \frac{3\pi}{2}$, $k \in \mathbb{Z}$

2) $\displaystyle\int \sin^3(x)\,dx \xrightarrow[du=-\sin(x)dx]{u=\cos(x)} \int \sin^2(x)\cdot\sin(x)\,dx = \int (1-u^2)(-du) = \int (u^2-1)\,du = \frac{u^3}{3} - u + C$

$\displaystyle = \frac{1}{3}\cos^3(x) - \cos(x) + C \quad , \quad x\in\mathbb{R}.$

3) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx \xrightarrow[du=-\sin x\,dx]{u=\cos(x)} \int \sin^2(x)\cos^2(x)\sin(x)\,dx = \int (1-u^2)u^2(-du) = \int (u^4-u^2)\,du =$

$\displaystyle = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$

4) $\displaystyle\int \frac{dx}{[1+\sin^2(x)]} \xrightarrow[\substack{dx=\frac{du}{1+u^2} \\ \cos^2(x)=\frac{1}{1+u^2}}]{u=\tan(x),\ x=\mathrm{Arctan}(u)} \int \frac{1}{\left[1+\left(1-\frac{1}{1+u^2}\right)\right]}\,\frac{du}{1+u^2} = \int \frac{du}{\frac{2u^2+1}{1+u^2}}\cdot\frac{1}{1+u^2} =$

$\displaystyle = \int \frac{du}{2u^2+1} = \frac{1}{2}\int \frac{du}{u^2+1/2} = \frac{1}{2}\sqrt{2}\,\mathrm{Arctan}(u\sqrt{2}) + C =$

$\displaystyle = \frac{1}{\sqrt{2}}\,\mathrm{Arctan}\left[\sqrt{2}\tan(x)\right] + C \quad , \quad \forall x\in I$, I διάστημα χωρίς ρίζες του cos.

5) $\displaystyle\int \frac{dx}{\cos^3(x)} \xrightarrow[du=\cos(x)dx]{u=\sin(x)} \int \frac{\cos(x)\,dx}{\cos^4(x)} = \int \frac{du}{(1-u^2)^2} = \int \frac{du}{(u-1)^2(u+1)^2}$

$\displaystyle \frac{1}{(u-1)^2(u+1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{\Gamma}{u+1} + \frac{\Delta}{(u+1)^2} \quad , \quad \forall u\neq 1,-1.$

$\displaystyle \Rightarrow \frac{1}{(u+1)^2} = B + A(u-1) + \frac{\Gamma(u-1)^2}{u+1} + \frac{\Delta(u-1)^2}{(u+1)^2} \ , \ \forall u\neq -1,1 \xRightarrow{u\to 1} B = \frac{1}{4}.$

Επίσης, $\displaystyle \frac{1}{(u-1)^2} = \frac{A(u+1)^2}{u-1} + \frac{B(u+1)^2}{(u-1)^2} + \Gamma(u+1) + \Delta \ , \ \forall u\neq -1,1 \xRightarrow{u\to -1} \Delta = 1/4$

Επίσης, $\displaystyle \frac{u-1}{(u-1)^2(u+1)^2} = A + \frac{B}{u-1} + \frac{\Gamma(u-1)}{u+1} + \frac{\Delta(u-1)}{(u+1)^2} \ , \ \forall u\neq -1,1 \xRightarrow{u\to +\infty} 0 = A+\Gamma$

Επίσης, για $u=0 \Rightarrow 1 = -A+B+\Gamma+\Delta \Rightarrow \Gamma - A + \frac{1}{2} = 1 \Rightarrow \frac{1}{2} = \Gamma - A$

$\Rightarrow \Gamma = 1/4$, $A = -1/4$

Άρα, $\displaystyle \frac{1}{(u-1)^2(u+1)^2} = -\frac{1}{4}\frac{1}{u-1} + \frac{1}{4}\frac{1}{(u-1)^2} + \frac{1}{4}\frac{1}{u+1} + \frac{1}{4}\frac{1}{(u+1)^2} \ , \ \forall u\neq 1,-1. \Rightarrow$

$\displaystyle \Rightarrow \int \frac{du}{(u-1)^2(u+1)^2} = -\frac{1}{4}\ln|u-1| - \frac{1}{4}\frac{1}{u-1} + \frac{1}{4}\ln|u+1| - \frac{1}{4}\frac{1}{u+1} + C \Rightarrow \int \frac{dx}{\cos^3(x)} = \frac{1}{4}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + \frac{1}{4(1-\sin x)} - \frac{1}{4(1+\sin x)} + C$

$\forall x\in I$, I διάστημα που δεν περιέχει τους $2k\pi+\pi/2,\ 2k\pi+\frac{3\pi}{2}$, $\forall k\in\mathbb{Z}$

6) $\int \frac{\sin^3(x)}{5+\cos^2(x)}dx \underset{du=-\sin(x)dx}{\overset{u=\cos(x)}{=\!=\!=}} \int \frac{\sin^2(x)\cdot\sin(x)}{5+\cos^2(x)}dx = \int\frac{(1-u^2)(-1)du}{5+u^2} = \int \frac{u^2-1}{u^2+5}du =$

$= \int \frac{u^2+5-6}{u^2+5}du = \int\left(1-\frac{6}{u^2+5}\right)du = u - 6\int\frac{du}{u^2+5} = u - \frac{6}{\sqrt5}\text{Arctan}\left(\frac{u}{\sqrt5}\right)+C =$

$= \cos(x) - \frac{6}{\sqrt5}\text{Arctan}\left(\frac{\cos(x)}{\sqrt5}\right)+C$

7) $\int\frac{\cos(x)}{\sin(x)+2\tan(x)}dx = \int\frac{\cos(x)}{\sin(x)+\frac{2\sin(x)}{\cos(x)}}dx = \int\frac{\cos^2(x)}{\sin(x)\cos(x)+2\sin(x)}dx \underset{du=-\sin x dx}{\overset{u=\cos(x)}{=\!=\!=}} \int\frac{\cos^2(x)\sin(x)}{\sin^2(x)\cos(x)+2\sin^2(x)}dx$

$= \int\frac{u^2(-1)du}{(1-u^2)u+2(1-u^2)} = \int\frac{u^2}{(u^2-1)(u+2)}du$

$\frac{u^2}{(u^2-1)(u+2)} = \frac{A}{u-1}+\frac{B}{u+1}+\frac{\Gamma}{u+2}, \quad \forall u\neq 1,-1,2.$

$A = \lim_{u\to1}\frac{u^2}{(u+1)(u+2)} = \frac{1}{3\cdot2} = \frac16. \quad B = \lim_{u\to-1}\frac{u^2}{(u-1)(u+2)} = -\frac12, \quad \Gamma = \lim_{u\to-2}\frac{u^2}{u^2-1} = \frac43$

$\Rightarrow \int\frac{u^2}{(u^2-1)(u+2)}du = \frac16\ln|u-1| - \frac12\ln|u+1| + \frac43\ln|u+2| = \frac16\ln(1-\cos(x)) - \frac12\ln(1+\cos(x)) + \frac43\ln(2+\cos x)+C$

$\forall x\in I$, I διάστημα χωρίς ρίζες του sin.

8) $\int\frac{\sin(x)}{1+\cos^2(x)}dx \underset{du=-\sin x dx}{\overset{u=\cos x}{=\!=\!=}} \int\frac{-du}{1+u^2} = -\text{Arctan}(u)+C = -\text{Arctan}(\cos(x))+C, \quad x\in\mathbb{R}$

9) $\int\frac{\cos(x)}{\sin^3(x)-\cos^3(x)}dx = \int\frac{\cos(x)}{\cos^3(x)\left[\tan^3(x)-1\right]}dx = \int\frac{dx}{\cos^2(x)\left[\tan^3(x)-1\right]} \underset{du=\frac{dx}{\cos^2(x)}}{\overset{u=\tan(x)}{=\!=\!=}} \int\frac{du}{u^3-1} \underset{\text{Αορ. III}}{\overset{\text{Άσκ. 5}}{=\!=\!=}}$

$= \frac13\ln|u-1| - \frac16\ln(u^2+u+1) - \frac13\text{Arctan}\left(\frac{2u+1}{\sqrt3}\right)+C = \frac13\ln|\tan(x)-1| - \frac16\ln(\tan^2x+\tan x+1) -$

$-\frac13\text{Arctan}\left(\frac{2\tan x+1}{\sqrt3}\right)+C, \quad \forall x\in I$, I διάστημα που δεν περιέχει ρίζες

του cos και ρίζες της $\tan(x)=1$ $\left(2k\pi+\pi/4,\ k\in\mathbb{Z}\right)$

10) $\int \cos^4(x)dx = \int \cos^2(x)\cos^2(x)dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \int \frac{1+\cos^2(2x)+2\cos(2x)}{4}dx =$

$= \int\left(\frac{1}{4} + \frac{1}{4}\cos^2(2x) + \frac{1}{2}\cos(2x)\right)dx = \frac{x}{4} + \frac{1}{4}\sin(2x) + \frac{1}{4}\int \cos^2(2x)dx = \frac{x}{4} + \frac{\sin(2x)}{4} +$

$+ \frac{1}{4}\int \frac{1+\cos(4x)}{2}dx = \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{1}{32}\sin(4x) + C = \frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$

$(x \in \mathbb{R})$

Για το μετ/μο $t = \tan\left(\frac{x}{2}\right)$, $-\pi < x < \pi$, έχουμε τα ακόλουθα:

$\frac{x}{2} = \mathrm{Arctan}(t) \Rightarrow x = 2\,\mathrm{Arctan}(t) \Rightarrow \boxed{dx = \frac{2}{1+t^2}dt}$

Επίσης, $\cos^2(x/2) = \frac{1}{1+\tan^2(x/2)} = \frac{1}{1+t^2} \xRightarrow{\cos(x/2)>0} \cos(x/2) = \frac{1}{\sqrt{1+t^2}}$

$\Rightarrow \sin^2(x/2) = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2} \Rightarrow |\sin(x/2)| = \frac{|t|}{\sqrt{1+t^2}}$. Αν $\sin(x/2) \geq 0 \Rightarrow$

$\xRightarrow{\cos(x/2)>0} \tan(x/2) = t \geq 0$, και $\sin(x/2) \leq 0 \Rightarrow t \leq 0$. Άρα, $\sin(x/2) = \frac{t}{\sqrt{1+t^2}}$

$\Rightarrow \sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2t}{1+t^2} \Rightarrow \boxed{\sin(x) = \frac{2t}{1+t^2}}$.

Τέλος, $\cos(x) = \cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2(x/2) - 1 = 2\frac{1}{1+t^2} - 1 \Rightarrow \boxed{\cos(x) = \frac{1-t^2}{1+t^2}}$

Παραδείγματα: 1) $\int \frac{dx}{1+\sin x+\cos x}$, $0 < x < \frac{\pi}{2}$. Έχουμε, με $t = \tan(x/2)$, ότι

$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{1+t^2+2t+1-t^2} = \int \frac{2dt}{2+2t} = \int \frac{dt}{1+t} = \ln|1+t| + C =$

$= \ln|1+\tan(x/2)| + C = \ln\left(1+\tan\left(\frac{x}{2}\right)\right) + C$, $0 < x < \pi/2$.

2) $\int \frac{dx}{2+\sin(x)}$, $-\pi < x < \pi$. Για $t = \tan(x/2) \Rightarrow \int \frac{dx}{2+\sin x} = \int \frac{\frac{2dt}{1+t^2}}{2+\frac{2t}{1+t^2}} =$

$= \int \frac{2dt}{2+2t^2+2t} = \int \frac{dt}{1+t^2+t} = \int \frac{dt}{t^2+2\cdot t\cdot\frac{1}{2}+\frac{1}{4}+\frac{3}{4}} = \int \frac{dt}{(t+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$

$= \frac{2}{\sqrt{3}}\mathrm{Arctan}\left(\frac{2t+1}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\mathrm{Arctan}\left[\frac{2\tan(x/2)+1}{\sqrt{3}}\right] + C$, $-\pi < x < \pi$.

3) $\displaystyle\int \frac{1+\sin(x)}{1+\cos(x)}\,dx$, $-\pi<x<\pi$. Θέτουμε $t=\tan(x/2)$. Τότε,

$$\int \frac{1+\sin x}{1+\cos x}\,dx = \int \frac{1+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\,\frac{2dt}{1+t^2} = \int \frac{1+t^2+2t}{1+t^2+1-t^2}\,\frac{2dt}{1+t^2} = \int \frac{1+t^2+2t}{1+t^2}\,dt =$$

$$= \int\left(1+\frac{2t}{1+t^2}\right)dt = \int dt + \int \frac{2t}{1+t^2}\,dt = t+\ln(1+t^2)+C = \tan\left(\frac{x}{2}\right)+\ln\left(1+\tan^2\left(\frac{x}{2}\right)\right)+C,$$

$-\pi<x<\pi$.

4) $$\int \frac{1+\sin x}{2+\cos x}\,dx \xrightarrow[(-\pi<x<\pi)]{t=\tan(x/2)} \int \frac{1+\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\,\frac{2dt}{1+t^2} = \int \frac{1+t^2+2t}{2+2t^2+1-t^2}\,\frac{2dt}{1+t^2} =$$

$$= 2\int \frac{1+t^2+2t}{(3+t^2)(1+t^2)}\,dt = 2\int \frac{1+t^2}{(3+t^2)(1+t^2)}\,dt + 2\int \frac{2t}{(3+t^2)(1+t^2)}\,dt = 2\int \frac{dt}{t^2+3} + 4\int \frac{t}{(3+t^2)(1+t^2)}\,dt =$$

$$= 2\,\frac{1}{\sqrt{3}}\,\mathrm{Arctan}\left(\frac{t}{\sqrt{3}}\right) + 4\int \frac{t}{(t^2+1)(t^2+3)}\,dt$$

Ανάλυση του $\dfrac{t}{(t^2+1)(t^2+3)}$ σε απλά κλάσματα: Κάλλιο $= \dfrac{1}{(s+1)(s+3)} = \dfrac{A}{s+1}+\dfrac{B}{s+3} \Rightarrow$

$$\Rightarrow \frac{1}{(s+1)(s+3)} = \frac{1}{2}\,\frac{1}{s+1} - \frac{1}{2}\,\frac{1}{s+3},\ \forall s\neq -1,-3. \Rightarrow \frac{t}{(t^2+1)(t^2+3)} = \frac{1}{2}\,\frac{t}{t^2+1} - \frac{1}{2}\,\frac{t}{t^2+3}$$

Δεν είναι ανάλυση σε απλά κλάσματα, όμως τα ολοκληρώματα που προκύπτουν υπολογίζονται εύκολα: $\displaystyle\int \frac{t\,dt}{(t^2+1)(t^2+3)} = \frac{1}{2}\int\frac{t\,dt}{t^2+1} - \frac{1}{2}\int \frac{t\,dt}{t^2+3} = \frac{1}{4}\ln(1+t^2)-\frac{1}{4}\ln(3+t^2)+C$

$$\Rightarrow \int \frac{1+\sin x}{2+\cos x}\,dx = \frac{2}{\sqrt{3}}\,\mathrm{Arctan}\left[\frac{\tan(x/2)}{\sqrt{3}}\right] + \ln\left[\frac{1+\tan^2(x/2)}{3+\tan^2(x/2)}\right]+C,\quad -\pi<x<\pi$$

Άσκηση: $\displaystyle\int\sqrt{6x-x^2-8}\,dx = \int\sqrt{6x-x^2-9+1}\,dx = \int\sqrt{1-(x-3)^2}\,dx$. Πρέπει λοιπόν $(x-3)^2\leq 1 \Leftrightarrow |x-3|\leq 1 \Leftrightarrow 2\leq x\leq 4$

Θέτουμε $x-3=\sin(t)$, $-\frac{\pi}{2}<t<\frac{\pi}{2}$. Τότε $\sqrt{1-(x-3)^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=\cos t$, αφού $-\frac{\pi}{2}<t<\pi/2$.

$$\Rightarrow \int\sqrt{1-(x-3)^2}\,dx = \int \cos t\cdot\cos t\,dt = \int\cos^2 t\,dt = \int\frac{1+\cos(2t)}{2}\,dt = \frac{t}{2}+\frac{1}{4}\sin(2t)+C = \frac{1}{2}\mathrm{Arcsin}(x-3)+\frac{1}{2}\sin t\cos t+C$$

$$= \frac{1}{2}\mathrm{Arcsin}(x-3)+\frac{1}{2}(x-3)\sqrt{1-(x-3)^2}+C = \frac{1}{2}\mathrm{Arcsin}(x-3)+\frac{1}{2}(x-3)\sqrt{6x-x^2-8}+C,\quad 2<x<4.$$