

Algorithmic Game Theory

Algorithms for 0-sum games

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Nash equilibria: Computation

- Nash's theorem only guarantees the existence of Nash equilibria
 - Proof via Brouwer's fixed point theorem
- The proof does not imply an efficient algorithm for computing equilibria
 - Because we do not have efficient algorithms for finding fixed points of continuous functions
- Can we design polynomial time algorithms for 2-player games?
 - For games with more players?

Zero-sum Games

A special case: 0-sum games

- Games where for every profile (s_i, t_j) we have

$$u_1(s_i, t_j) + u_2(s_i, t_j) = 0$$

- The payoff of one player is the payment made by the other
- Also referred to as **strictly competitive**
- It suffices to use only the matrix of player 1 to represent such a game
- How should we play in such a game?

4	2
1	3

A special case: 0-sum games

- **Idea:** Pessimistic play
- Assume that no matter what you choose the other player will pick the worst outcome for you
- Reasoning of player 1:
 - If I pick row 1, in worst case I get 2
 - If I pick row 2, in worst case I get 1
 - I will pick the row that has the best worst case
 - Payoff = $\max_i \min_j A_{ij} = 2$
- Reasoning of player 2:
 - If I pick column 1, in worst case I pay 4
 - If I pick column 2, in worst case I pay 3
 - I will pick the column that has the smallest worst case payment
 - Payment = $\min_j \max_i A_{ij} = 3$

4	2
1	3

0-sum games

Definitions

- For pl. 1:
 - The best of the worst-case scenarios:
$$v_1 = \max_i \min_j A_{ij}$$
 - We take the minimum of each row and select the best minimum
- For pl. 2:
 - Again the best of the worst-case scenarios
$$v_2 = \min_j \max_i A_{ij}$$
 - We take the max in each column and then select the best maximum
- In the example:
 - $v_1 = 2, v_2 = 3$
- The game also does not have pure Nash equilibria

Example 2

- Computing v_1 for pl. 1:
 - Row 1, min = 4
 - Row 2, min = 1
 - Row 3, min = 0
 - Row 4, min = 4
 - $v_1 = \max \{4, 1, 0, 4\} = 4$
- Computing v_2 for pl. 2:
 - Column 1, max = 4
 - Column 2, max = 6
 - Column 3, max = 7
 - Column 4, max = 4
 - $v_2 = \min \{4, 6, 7, 4\} = 4$

	t_1	t_2	t_3	t_4
s_1	4	5	6	4
s_2	2	6	1	3
s_3	1	0	0	2
s_4	4	4	7	4

Example 2

- In contrast to the first example, here we have $v_1 = v_2$
- Recommended strategies:
 - s_1 or s_4 for pl. 1
 - t_1 or t_4 for pl. 2
- Pessimistic play can lead to 4 different profiles
- Observations:
 - i. Same utility in all 4 profiles
 - ii. All 4 profiles are Nash equilibria!
 - iii. There is no other Nash equilibrium

	t_1	t_2	t_3	t_4
s_1	4	5	6	4
s_2	2	6	1	3
s_3	1	0	0	2
s_4	4	4	7	4

Nash equilibria in 0-sum games

Theorem: For every finite 2-player 0-sum game:

- $v_1 \leq v_2$
- There exists a Nash equilibrium with pure strategies if and only if $v_1 = v_2$
- If (s, t) and (s', t') are pure equilibria, then the profiles (s, t') , (s', t) are also equilibria
- When we have multiple Nash equilibria, the utility is the same for both players in all equilibria (v_1 for pl. 1 and $-v_1$ for pl. 2)

Corollary: In games where $v_1 < v_2$, there is no Nash equilibrium with pure strategies

Nash equilibria in 0-sum games

- In general $v_1 \neq v_2$
- Pessimistic play with pure strategies does not always lead to a Nash equilibrium
- **Idea (von Neumann):** Use pessimistic play with mixed strategies!
- Definitions:
 - $w_1 = \max_{\mathbf{p}} \min_{\mathbf{q}} u_1(\mathbf{p}, \mathbf{q})$
 - $w_2 = \min_{\mathbf{q}} \max_{\mathbf{p}} u_1(\mathbf{p}, \mathbf{q})$
- We can easily show that: $v_1 \leq w_1 \leq w_2 \leq v_2$
 - Because we are optimizing over a larger strategy space
- How can we compute w_1 and w_2 ?

Back to Example 1

- We will find first $w_1 = \max_{\mathbf{p}} \min_{\mathbf{q}} u_1(\mathbf{p}, \mathbf{q})$
- We need to look for a strategy $\mathbf{p} = (p_1, p_2) = (p_1, 1 - p_1)$ of pl. 1
- We need to look better at the 2 consecutive optimization steps
- **Lemma**: Given a strategy \mathbf{p} of pl. 1, the term $\min_{\mathbf{q}} u_1(\mathbf{p}, \mathbf{q})$ is minimized at a pure strategy of pl. 2
 - Hence, no need to have both optimization steps over mixed strategies

4	2
1	3

Analysis of Example 1

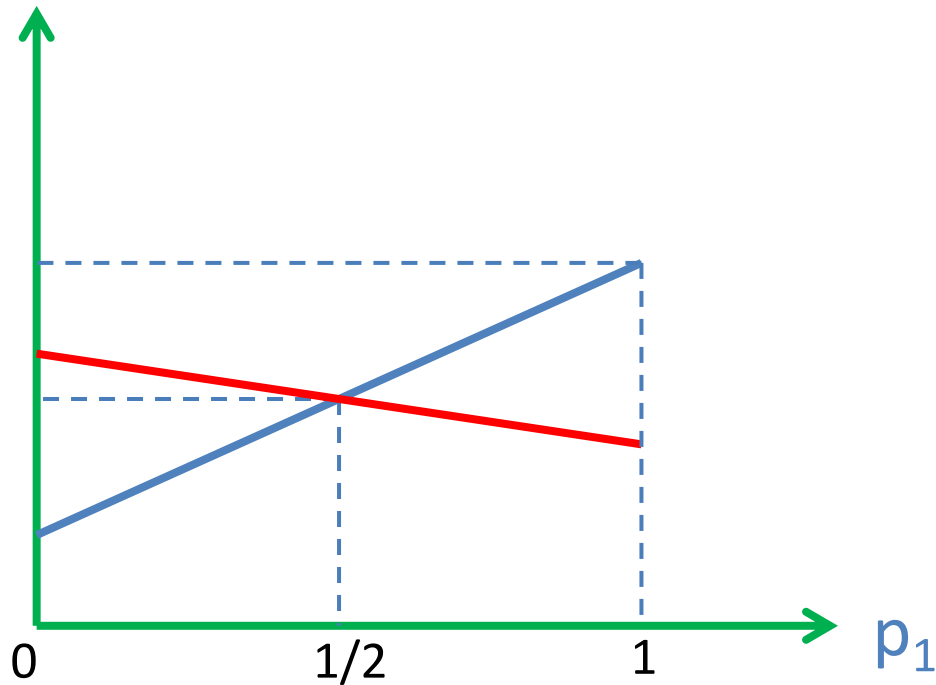
- The lemma simplifies the process as follows:

$$\begin{aligned}w_1 &= \max_{\mathbf{p}} \min_{\mathbf{q}} u_1(\mathbf{p}, \mathbf{q}) \\ &= \max_{\mathbf{p}} \min\{ u_1(\mathbf{p}, e^1), u_1(\mathbf{p}, e^2) \} \\ &= \max_{p_1} \min\{ 4p_1 + 1 - p_1, 2p_1 + 3(1 - p_1) \} \\ &= \max_{p_1} \min\{ 3p_1 + 1, 3 - p_1 \}\end{aligned}$$

4	2
1	3

Analysis of Example 1

- $w_1 = \max_{p_1} \min \{ 3p_1 + 1, 3 - p_1 \}$
- We need to maximize the minimum of 2 lines

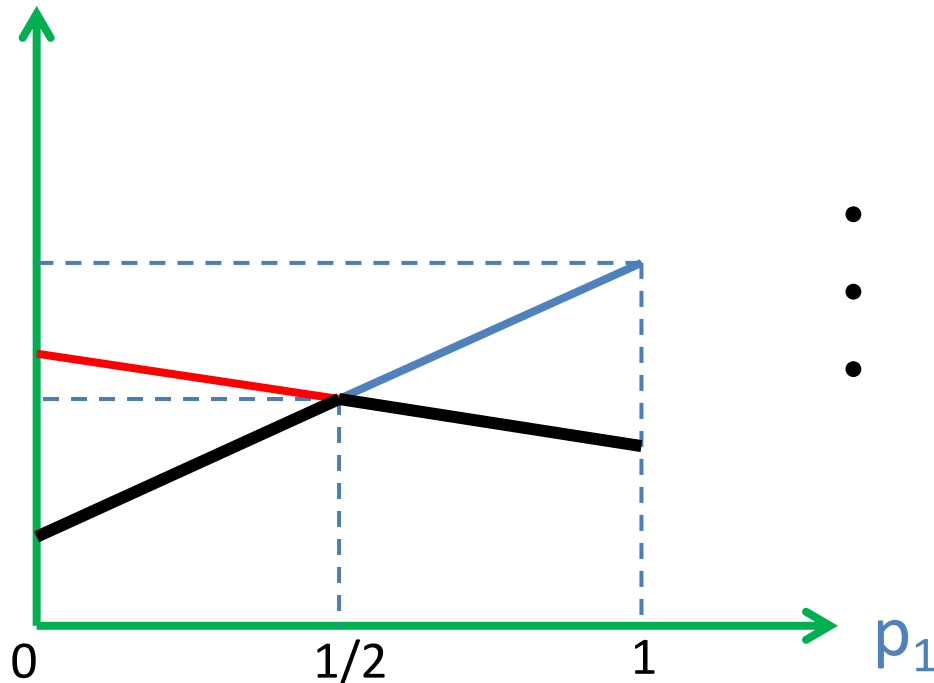


4	2
1	3

Analysis of Example 1

- $w_1 = \max_{p_1} \min \{ 3p_1 + 1, 3 - p_1 \}$
- We need to maximize the minimum of 2 lines

4	2
1	3



- One line is increasing
- The other is decreasing
- The min. is achieved at the intersection point $\rightarrow p_1 = 1/2$

Analysis of Example 1

Summing up:

- $w_1 = \max_{\mathbf{p}} \min_{\mathbf{q}} u_1(\mathbf{p}, \mathbf{q}) = \max_{p_1} \min \{ 3p_1 + 1, 3 - p_1 \} = 3 \cdot 1/2 + 1 = 5/2$
- If pl. 1 plays strategy $\mathbf{p} = (1/2, 1/2)$, he can guarantee on average $5/2$, independent of the choice of pl. 2
- Thus, with mixed strategies, pessimistic play provides a better guarantee than with pure ($v_1 = 2 < 2.5$)

4	2
1	3

Analysis of Example 1

With a similar analysis for pl. 2:

$$\begin{aligned}w_2 &= \min_{\mathbf{q}} \max_{\mathbf{p}} u_1(\mathbf{p}, \mathbf{q}) \\ &= \min_{\mathbf{q}} \max\{ u_1(e^1, \mathbf{q}), u_1(e^2, \mathbf{q}) \} \\ &= \min_{q_1} \max\{ 4q_1 + 2(1-q_1), q_1 + 3(1-q_1) \} \\ &= \min_{q_1} \max\{ 2q_1 + 2, 3 - 2q_1 \}\end{aligned}$$

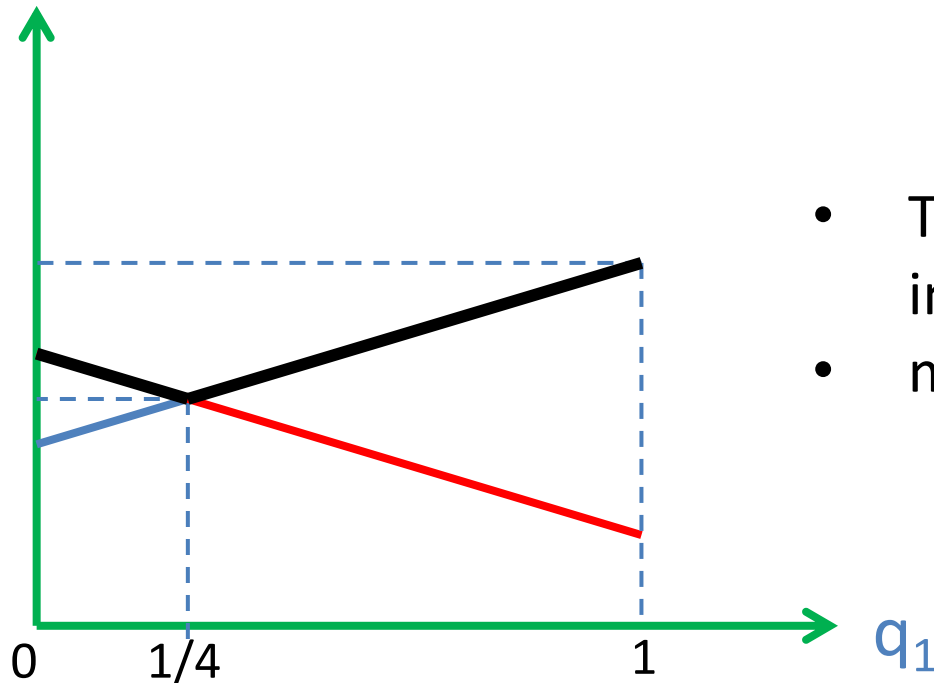
4	2
1	3

- We now want to minimize the max among 2 lines

Analysis of Example 1

- $w_2 = \min_{q_1} \max\{ 2q_1 + 2, 3 - 2q_1 \}$
- Again, one is increasing, the other is decreasing

4	2
1	3



- The max. is achieved at the intersection point $\rightarrow q_1 = 1/4$
- min-max strategy: $(1/4, 3/4)$

Analysis of Example 1

Final conclusions:

- We found the profile
 - $\mathbf{p} = (1/2, 1/2)$, $\mathbf{q} = (1/4, 3/4)$
- $w_1 = w_2 = 5/2$
- Both players guarantee something better to themselves by using mixed strategies
- With pure strategies:
$$\max_i \min_j A_{ij} \neq \min_j \max_i A_{ij}$$
- With mixed strategies, we have equality
$$\max_{\mathbf{p}} \min_{\mathbf{q}} u_1(\mathbf{p}, \mathbf{q}) = \min_{\mathbf{q}} \max_{\mathbf{p}} u_1(\mathbf{p}, \mathbf{q})$$
- Also, (\mathbf{p}, \mathbf{q}) is a Nash equilibrium! (check)

4	2
1	3

Nash equilibria in 0-sum games

Theorem (von Neumann, 1928): For every finite 2-player 0-sum game:

1. $w_1 = w_2$ (referred to as the **value** of the game)
2. The profile (\mathbf{p}, \mathbf{q}) , where w_1 and w_2 are achieved forms a Nash equilibrium
3. If (\mathbf{p}, \mathbf{q}) and $(\mathbf{p}', \mathbf{q}')$ are equilibria, then the profiles $(\mathbf{p}, \mathbf{q}')$, $(\mathbf{p}', \mathbf{q})$ are also equilibria .
4. In every Nash equilibrium, the utility to each player is the same (w_1 for pl. 1 and $-w_1$ for pl. 2)

Nash equilibria in 0-sum games

Conclusions from von Neumann's theorem

- For the family of 2-player 0-sum games, all the problematic issues we had identified for normal form games are resolved
 - Existence: guaranteed
 - Non-uniqueness: not a problem, because all equilibria yield the same utility to each player
 - If there are multiple equilibria, all of them are equally acceptable

Nash equilibria in 0-sum games

Computation of Nash equilibria

- Till now we saw how to find Nash equilibria in 2×2 0-sum games
- The same reasoning can also be applied for $2 \times n$ games
- Can we find an equilibrium for arbitrary $n \times m$ 0-sum games?

0-sum nxm games

- What happens when $n \geq 3$ and $m \geq 3$?
- With 4 pure strategies, we need to look for a mixed strategy of pl. 1 in the form $\mathbf{p} = (p_1, p_2, p_3, 1 - p_1 - p_2 - p_3)$
- Suppose we start with the same methodology:

	t_1	t_2	t_3	t_4
s_1	6	5	3	5
s_2	1	2	6	4
s_3	3	8	3	2
s_4	5	4	2	0

$$\begin{aligned}
 w_1 &= \max_{\mathbf{p}} \min_{\mathbf{q}} u_1(\mathbf{p}, \mathbf{q}) \\
 &= \max_{\mathbf{p}} \min\{ u_1(\mathbf{p}, e^1), u_1(\mathbf{p}, e^2), u_1(\mathbf{p}, e^3), u_1(\mathbf{p}, e^4) \} \\
 &= \max_{p_1, p_2, p_3} \min\{ 6p_1 + p_2 + 3p_3 + 5(1 - p_1 - p_2 - p_3), 5p_1 + 2p_2 + 8p_3 + 4(1 - p_1 - p_2 - p_3), \dots, \dots \}
 \end{aligned}$$

- Problem with 3 variables, cannot visualize as before

0-sum nxm games

- We need a different approach
- We can try to see if von Neumann's theorem implies an efficient algorithm
- The initial proof of von Neumann's theorem (1928) is not constructive
 - Based on fixed point theorems
- **Fortunately:** there is an alternative algorithmic proof of existence
- Finding w_1 and the strategy of pl. 1 can be modeled as a linear programming problem
- Finding the equilibrium strategy of pl. 2 can be modeled as the **dual** problem to that of pl. 1

Linear Programming

- What is a linear program?
- Any optimization problem where
 - The objective function is linear
 - The constraints are also linear

$$\text{maximize } Z(x) = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

⋮

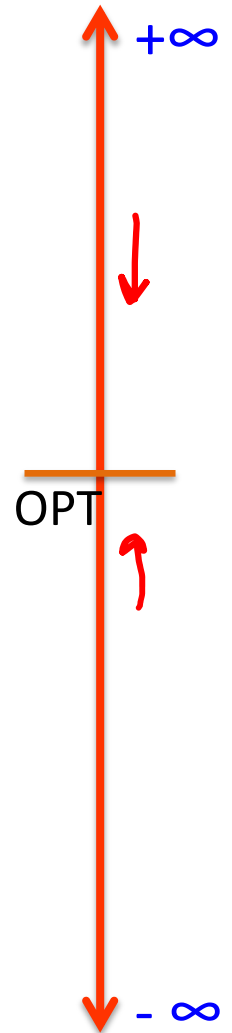
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

- We can also have inequalities with \geq or equalities in the constraints
- We can solve linear programs very fast, even with hundreds of variables and constraints (Matlab, AMPL,...)

Linear Programming

- Basic component for the alternative proof of von Neumann's theorem:
- **Duality theorem:** For every maximization LP, there is a corresponding dual minimization LP such that
 - The primal LP has an optimal solution iff the dual LP has an optimal solution
 - The optimal value (when it exists) for both the primal and the dual LP is the same



Nash equilibria in 0-sum games

- Consider a 0-sum game with an $n \times m$ matrix A for pl. 1
- **LP-based proof of von Neumann's theorem:** The max-min and the min-max strategies of pl. 1 and pl. 2 are obtained by solving the linear programs:

$$\begin{aligned}
 & \max w \\
 & \text{s. t.:} \\
 & w \leq \sum_{i=1}^n A_{ik} p_i, \forall k = 1, \dots, m \\
 & \sum_{i=1}^n p_i = 1 \\
 & p_i \geq 0, \quad \forall i = 1, \dots, n
 \end{aligned}$$

$u_i(p, e^k)$

Primal LP

$$\begin{aligned}
 & \min w \\
 & \text{s. t.:} \\
 & w \geq \sum_{j=1}^m A_{ij} q_j, \forall i = 1, \dots, n \\
 & \sum_{j=1}^m q_j = 1 \\
 & q_j \geq 0, \quad \forall j = 1, \dots, m
 \end{aligned}$$

Dual LP

Example

- $v_1 = 3, v_2 = 5$, no pure Nash equilibrium
- We have to use linear programming to find the equilibrium profile

Primal LP

max w

s.t.

$$w \leq 6p_1 + p_2 + 3p_3$$

$$w \leq 5p_1 + 2p_2 + 8p_3$$

$$w \leq 3p_1 + 6p_2 + 3p_3$$

$$w \leq 5p_1 + 4p_2 + 2p_3$$

$$p_1 + p_2 + p_3 = 1$$

$$p_1, p_2, p_3 \geq 0$$

	t_1	t_2	t_3	t_4
s_1	6	5	3	5
s_2	1	2	6	4
s_3	3	8	3	2

Dual LP

min w

s.t.

$$w \geq 6q_1 + 5q_2 + 3q_3 + 5q_4$$

$$w \geq q_1 + 2q_2 + 6q_3 + 4q_4$$

$$w \geq 3q_1 + 8q_2 + 3q_3 + 2q_4$$

$$q_1 + q_2 + q_3 + q_4 = 1$$

$$q_1, q_2, q_3, q_4 \geq 0$$

Summary on 0-sum games

- There always exists a Nash equilibrium in finite 0-sum games, when we allow mixed strategies
- $w_1 = w_2 =$ value of the game
- If there are multiple equilibria, they all have the same utility for each player (w_1 for pl. 1, $-w_1$ for pl. 2)
- The value of the game as well as the equilibrium profile can be computed in polynomial time by solving a pair of primal and dual linear programs

0-sum games and optimization

Further connections with Computer Science and Algorithms:

1. Every linear program is “**equivalent**” to solving a 0-sum game
 - Finding the optimal solution to any linear program can be reduced to finding an equilibrium in some 0-sum game
 - Initially stated in [Dantzig '51], complete proof in [Adler '13]
2. Every problem solvable in polynomial time (class **P**), can be reduced to linear programming, and hence to finding a Nash equilibrium in some appropriately constructed 0-sum game!

0-sum games and complexity classes

Class **P**

Shortest paths,
minimum spanning
trees, sorting, ...



0-sum games

Matching Pennies,
Rock-Paper-Scissors,
...

And some more observations

- Anything we have seen so far also hold for **constant-sum games**
- In a constant-sum game, for every profile (s, t) with $s \in S^1$, $t \in S^2$
 $u_1(s, t) + u_2(s, t) = c$, for some parameter c
- **WHY?**
 - We can subtract c from the payoff matrix of pl. 1 (or pl. 2 but not both), so as to convert it to a 0-sum game
 - Adding/subtracting the same parameter from every cell of a payoff matrix do not change the set of Nash equilibria