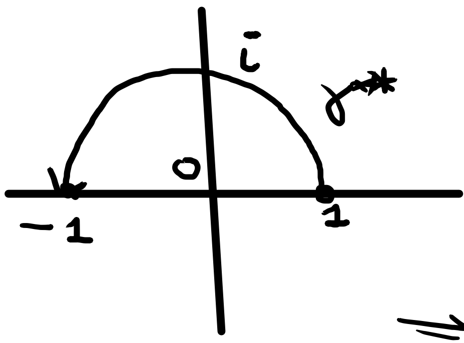
13/03/2024

Ασκήσεις (Φυλλ. Ασκήσεων Κεφ. 4-11)

Γ1 $\displaystyle\int_\gamma \overline{z^2 e^z}\, dz = ?$ $\quad \gamma(t) = e^{it}, \ t \in [0, \pi]$

Λύση:



$\forall \ z \in \gamma^*, \ |z| = 1$

$\Rightarrow \quad \bar{z} = 1/z$

$\Rightarrow \quad \overline{z^2 e^z} = \bar{z}^2 \cdot e^{\bar{z}}$

$= \dfrac{1}{z^2} e^{1/z} = -\left(e^{1/z}\right)'$

$\forall \ z \in U = \mathbb{C} \setminus \{0\}$

$\gamma^* \subset U = \mathbb{C} \setminus \{0\}$ = ανοικτό

κ΄ αν $F(z) = -e^{1/z}$, τότε

F παράγουσα της $f(z) = \frac{1}{z^2} e^{1/z}$ στο U.

Επιπλέον, η αρχή και το πέρας της γ είναι

$1, -1$ αντίστοιχα. Άρα,

$$\int_\gamma \overline{z^2 e^z}\, dz = \int_\gamma f(z)\, dz = F(-1) - F(1)$$

$$= -e^{-1} + e = \boxed{e - \frac{1}{e}}$$

(Π2) $\left|\int_\gamma \frac{e^{iz}}{z^2}\,dz\right| \leq \pi$, $\gamma(t) = e^{it}$, $t \in [0,\pi]$

Λύση: Έστω $z = x+iy \in \gamma^*$. Τότε,

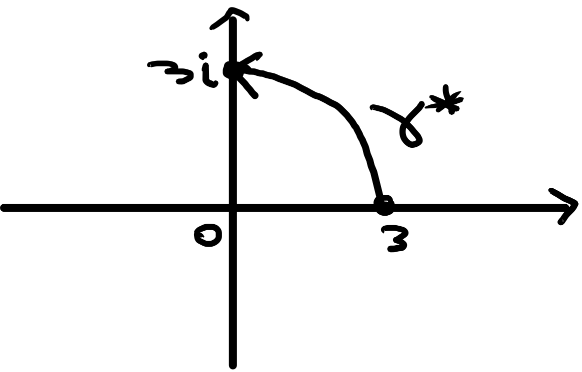
$iz = -y + ix \Rightarrow |e^{iz}| = e^{Re(iz)} = e^{-y}$

κ' $y \geq 0 \Rightarrow |e^{iz}| \leq 1 \Rightarrow \left|\frac{e^{iz}}{z^2}\right| =$

$= \frac{|e^{iz}|}{|z|^2} = |e^{iz}| \leq 1$, $\forall z \in \gamma^*$

$\overset{(M-L)}{\Longrightarrow} \left|\int_\gamma \frac{e^{iz}}{z^2}\,dz\right| \leq 1 \cdot \|\gamma\| = \pi.$

$$\left| \int_\gamma \frac{dz}{\bar{z}^2 + \bar{z} + 1} \right| \le \frac{3\pi}{10}, \quad \gamma(t) = 3e^{it}, \quad t \in [0, \pi/2]$$



$\forall z \in \gamma^*$,

$|\bar{z}^2 + \bar{z} + 1| \cdot$

$\ge |\bar{z}^2| - |\bar{z} + 1|$

$\ge |\bar{z}|^2 - |\bar{z}| - 1 =$

$(\ |\bar{z} + 1| \le |\bar{z}| + 1\ )$

$= |z|^2 - |z| - 1$

$= 3^2 - 3 - 1 = 5$

$$\Rightarrow \left| \frac{1}{\bar{z}^2 + \bar{z} + 1} \right| \le \frac{1}{5}, \forall z \in \gamma^* \quad \overset{(M-L)}{\Longrightarrow}$$

$$\Rightarrow \left| \int_\gamma \frac{dz}{z^2+z+1} \right| \leq \frac{1}{5} \|\gamma\| = \frac{1}{5} \cdot \frac{3\pi}{2} = \frac{3\pi}{10}.$$

$$\|\gamma\| = \frac{3 \cdot 2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$