SOLUTIONS PS 5

PROBLEM 1: Let $x \in L$ and let U_x be an open set containing x su[ch] that

$$f(x) < f(u) \quad \text{for all } u \in U_x \setminus \{x\}$$

Since X is separable it is second countable. Let $\mathcal{B}$ be a countable base. So, we can find $B_x \in \mathcal{B}$ such that

$$x \in B_x \subseteq U_x$$

Let $\xi : X \to \mathcal{B}$ be defined by

$$\xi(x) = B_x.$$

This map is injective. Indeed, arguing by contradiction suppose $x \neq u$ and $\xi(x) = \xi(u)$. Then

$$f(x) < f(u) \quad \text{and} \quad f(u) < f(x), \text{ contradiction.}$$

a contradiction. So L is at most countable.

QED

PROBLEM 2: Recall that $f(\cdot)$ is continuous iff it is both upper and lower semicontinuous, hence iff for all $\lambda \in \mathbb{R}$

$$\{f \geq \lambda\} \text{ is closed}, \ \{f > \lambda\} \text{ is open.}$$

QED

PROBLEM 3: $\Rightarrow$ We may assume that for some $x_0 \in X$ $f(x_0) < \infty$

Let

$$\hat{f}_n(x) = \inf\left[f(y) + n\,d(x,y) : y \in X\right]$$

$\xi \leq \hat{f}_n(x) \leq f(x)$ and $\xi \leq \hat{f}_n(x) \leq f(x) + d(x, x_0) < +\infty$

So, we have $\hat{f}_n(\circ)$ is $\mathbb{R}$-valued

$$\xi \leq \hat{f}_1 \leq \hat{f}_2 \leq \dots \leq \hat{f}_n \leq \dots$$

Using the triangle inequality we have

$$f(y) + n d(x,y) \leq f(y) + n d(y,u) + n d(u,x),$$

$$\Rightarrow \hat{f}_n(x) \leq \hat{f}_n(u) + n d(u,x).$$

Interchanging the roles of x and u in the above argument, we also have

$$\hat{f}_n(u) \leq \hat{f}_n(x) + n d(u,x),$$

$$\Rightarrow |\hat{f}_n(u) - \hat{f}_n(x)| \leq n d(u,x)$$

So $\hat{f}_n(\circ)$ is n-Lipschitz and

$$\lim \hat{f}_n(x) \leq f(x) \quad \forall x \in X$$

Given $\varepsilon > 0$, let $y_n \in X$ such that

$$f(y_n) + n d(y_n, x) \leq \hat{f}_n(x) + \varepsilon$$

As $n \to \infty$ either $\hat{f}_n(x) \uparrow +\infty$ and so $\hat{f}_n(x) \uparrow f(x) = \infty$ (since $\lim \hat{f}_n(x) \leq f(x)$) or else $d(y_n, x) \to 0$. Therefore

$$f(x) \leq \liminf_{n \to \infty} f(y_n) \leq \lim_{n \to \infty} \hat{f}_n(x) + \varepsilon,$$

$\Rightarrow f(x) \leq \lim_{n} \hat{f}(x)$ (let $\varepsilon \downarrow 0$)

$$\Rightarrow \hat{f}_n \uparrow f$$

Set $f_n = \min\{n, \hat{f}_n\}$. Then $f_n \in C_b(X)$ and

$$f_n \uparrow f$$

$\Leftarrow$ Supremum of continuous functions is lower semicontinuous.

QED

PROBLEM 4: Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of upper semicontinuous such that $f_n \xrightarrow{u} f$. Let $\lambda \in \mathbb{R}$ and $x \in \{f < \lambda\}$. We can find $\delta > 0$ such that

$$f_n(x) < \lambda - \delta \quad \forall n \geq n_1$$

The upper semicontinuity of each $f_n(\cdot)$ implies that we can find $\varepsilon_n > 0$ such that

$$f_n(y) < \lambda - \delta \quad \forall y \in (x - \varepsilon_n, x + \varepsilon_n)$$

Since $f_n \xrightarrow{u} f$, we can find $n_2 \in \mathbb{N}$, $n_2 \geq n_1$ such that

$$|f_n(u) - f(u)| < \delta \quad \forall n \geq n_2 \quad \forall u \in \mathbb{R},$$

$$\Rightarrow f_n(u) > f(u) - \delta \quad \forall n \geq n_2 \quad \forall u \in \mathbb{R},$$

$$\Rightarrow \lambda > f(u) \quad \forall u \in (x - \varepsilon_n, x + \varepsilon_n)$$

$$\Rightarrow \{\lambda > f\} \text{ is open and so } f \text{ is usc}$$

QED

PROBLEM 5: By hypothesis

$$d\left(f^{(k)}(x), f^{(k)}(u)\right) \leqslant c\, d(x,u) \quad 0<c<1,\ x,u \in X$$

$$\Rightarrow \exists!\ x \in X \text{ s.t } f^{(k)}(x) = x$$

(Banach fixed point theorem)

We have

$$d(f(x),x) = d\left(f\left(f^{(k)}(x)\right), f^{(k)}(x)\right)$$

$$= d\left(f^{(k)}(f(x)), f^{k}(x)\right)$$

$$\leqslant c\, d(f(x),x)$$

$$\Rightarrow d(f(x),x) = 0 \quad \text{since } c \in (0,1)$$

$$\Rightarrow x = f(x)$$

Finally if $f(u)=u$, then $f^{(k)}(u)=u$ and so from the uniqueness of x, we have $u=x$.

QED