# SOLUTIONS P.S. 4

**PROBLEM 1:** Let $\{x_i\}_{i\in I} \subseteq f^{-1}(K)$ be a net such that $x_i \to x$. Th

$$f(x_i) = y_i \in K \quad \text{for all } i \in I.$$

Since $K$ is compact we can find a subnet $\{y_j\}_{j\in J} \subseteq$ $\{y_i\}_{i\in I}$ such that

$$y_j \to y \in K.$$

We have $(x_j, y_j) \in Grf$ for all $j \in J$ and

$$(x_j, y_j) \to (x, y) \text{ in } X \times Y$$

$\Rightarrow (x,y) \in Grf$ with $y \in K$

$\Rightarrow x \in f^{-1}(K)$ and so the latter is closed

QED

**PROBLEM 2:** Let $x_n \to x$ in $H$ and $A(x_n) \to y$ in $H$. For every $u \in H$ we have

$$\begin{array}{ccc} (A(x_n), u) & = & (x_n, B(u)) \\ \downarrow & & \downarrow \\ (y, u) & & (x, B(u)), \end{array}$$

$$\Rightarrow (y, u) = (x, B(u)) = (A(x), u).$$

Since $u \in H$ is arbitrary, we infer that

$$y = A(x)$$

By the closed graph theorem $A \in \mathcal{L}(H)$. Similarly for $B$.

QED

PROBLEM 3: Let $y \in Y$, $\|y\| = 1$. Let $\beta_y : X \to Z$ be defined by

$$\beta_y(x) = \beta(x, y).$$

$$\Rightarrow \beta_y \in \mathcal{L}(X, Z).$$

So, there exists $M_y > 0$ such that

$$\|\beta_y(x)\| \leq M_y \|x\| \quad \text{for all } x \in X.$$

By the Banach-Steinhaus Theorem, there is $M > 0$ such that

$$\|\beta_y\|_{\mathcal{L}} \leq M,$$

$$\Rightarrow \|\beta_y(x)\| \leq M \|x\| \quad \forall x \in X,\ y \in Y \cap \partial B_1$$

The bilinearity of $\beta(\cdot, \cdot)$ implies

$$\|\beta(x, y)\| \leq M \|x\| \|y\|,$$

$$\Rightarrow \beta \text{ is jointly continuous.}$$

QED

PROBLEM 4: Since $\overline{D} = X$, given $x \in X$ we can find a net $\{x_i\}_{i \in I} \subseteq D$ such that $x_i \to x$. By hypothesis

$$f(x_i) = g(x_i) \quad \forall i \in I,$$

$$\Rightarrow f(x) = g(x) \quad (\text{by continuity}),$$

$$\Rightarrow f \equiv g \quad (\text{since } x \in X \text{ is arbitrary}).$$

QED

PROBLEM 5: $\Rightarrow$ Arguing by contradiction suppose $\overline{A(X)} \neq Y$

Then there exists $y^* \in Y^* \setminus \{0\}$ such that

$$y^*\big|_{A(X)} = 0,$$

$$\Rightarrow \langle y^*, A(x)\rangle = 0 \text{ for all } x \in X,$$

$$\Rightarrow \langle A^*(y^*), x\rangle = 0 \text{ for all } x \in X,$$

$$\Rightarrow A^*(y^*) = 0,$$

a contradiction since by hypothesis $A^*$ is 1-1

$\Leftarrow$ Let $A^*(y^*) = 0$ for some $y^* \in Y^*$ Then

$$\langle A^*(y^*), x\rangle = 0 \quad \forall x \in X,$$

$$\Rightarrow \langle y^*, A(x)\rangle = 0 \quad \forall x \in X$$

$$\Rightarrow y^*\big|_{A(X)} = 0,$$

$$\Rightarrow y^* = 0 \quad (\text{since } \overline{A(X)} = Y),$$

$$\Rightarrow A^* \text{ is 1-1.}$$

QED