SOLUTIONS PS 2.

PROBLEM 1: Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$u_n \longrightarrow u$ in X, $A(u_n) \longrightarrow u^*$ in X^*.

By hypothesis we have

$$\begin{array}{ccc} \langle A(u_n), h\rangle & = & \langle A(h), u_n\rangle \quad \forall n\in\mathbb{N}, \forall h\in X. \\ \downarrow & & \downarrow \\ \langle u^*, h\rangle & = & \langle A(h), u\rangle = \langle A(u), h\rangle \quad \forall h\in X, \end{array}$$

$\Rightarrow u^* = A(u)$,

$\Rightarrow A \in \mathcal{L}(X,Y)$ (by the closed graph theorem)

QED

PROBLEM 2: Let $V_1 = V \oplus \mathbb{R}u$ and let $h: V_1 \longrightarrow \mathbb{R}$ be the linear functional defined by

$$h(v+\lambda u) = \lambda\, d(u,V)$$

We see that $h(u) = d(u,V)$ and $h\big|_V = 0$. Also

$$|h(v+\lambda u)| = |\lambda|\, d(u,V) \leq |\lambda|\, \|u - (-\tfrac{v}{\lambda})\|$$
$$= \|v+\lambda u\|,$$

$\Rightarrow h \in V_1^*$ and $\|h\|_* \leq 1$

In addition we have

$$d(u,V) = |h(u-v)| \leq \|h\|_* \|u-v\| \quad \forall v \in V$$

$$\Rightarrow d(u,V) \leq \|h\|_* \, d(u,V).$$

$$\Rightarrow 1 \leq \|h\|_* \text{ and so } \|h\|_* = 1$$

Finally by the Hahn-Banach Theorem, we can find $u^* \in X^*$ such that $u^*\big|_{V_1} = h$ and $\|u^*\|_* = \|h\|_* = 1$

QED

PROBLEM 3: (a) $\Rightarrow$ (b)

For every $n \in \mathbb{N}$ C_n is w-closed and bounded. So C_n = w-compact (since X is reflexive). By the finite intersection property

$$\bigcap_{n\in\mathbb{N}} C_n \neq \phi$$

(b) $\Rightarrow$ (a)

Let $u^* \in X^* \setminus \{0\}$ and define

$$C_n = \left\{ u \in X : \|u\| \leq 1 \ , \ \langle u^*, u\rangle \geq \|u^*\|_* - \frac{1}{n} \right\}.$$

Then $\{C_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty, closed, convex, bounded sets in X. So, by hypothesis we have

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$$

Let $u \in \bigcap_{n\in\mathbb{N}} C_n$. Then

$$\|u\| \leq 1 \quad , \quad \langle u^*, u\rangle \geq \|u^*\|_* - \frac{1}{n} \quad \forall n \in \mathbb{N},$$

$$\Rightarrow \|u\| \leq 1 \, , \quad \langle u^*, u\rangle = \|u^*\|_*$$

Since u^* arbitrary by James' theorem

$$X = \text{Reflexive}$$

QED

PROBLEM 4. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ linearly independent set

$$V = \overline{\text{span}}\{u_n\}_{n\in\mathbb{N}},$$

$$\Rightarrow V = \text{Reflexive Separable},$$

$$\Rightarrow (\overline{B}_1^V, w) \text{ is compact metrizable.}$$

Also we know $\overline{\mathbb{Q} B_1^V}^w = \overline{B}_1^V$.

From this follows the result.

QED

PROBLEM 5: Let $j: X \to X^{**}$ be the canonical embedding, Let $V = j(X) \subseteq X^{**}$ and set $\hat{X} = \overline{V}$. This is a Banach space and V is dense in $\hat{X}$, while X is isometric, isomorphic to V.

QED