EXERCISES 4

Exercise 1: Let H be a separable Hilbert space. Is $\mathcal{L}(H)$ separable? Explain

Proof: Every separable Hilbert space is isomorphic to $L^2(0,1)$

So, we may assume that $H = L^2(0,1)$. Let

$$f_t = \chi_{(0,t)} \quad t \in (0,1)$$

Let $P_t \in \mathcal{L}(H)$ be defined by

$$P_t(u) = f_t u \quad \forall u \in H = L^2(0,1)$$

Then if $0<s<t<1$, we have

$$(P_t - P_s)(u) = \chi_{(s,t)} u \qquad \forall u \in L^2(0,1),$$

$$\Rightarrow \| (P_t - P_s)(u) \|_2^2 \leq \int_s^t u^2 dz \leq \| u \|_2^2,$$

$$\Rightarrow \| P_t - P_s \|_{\mathcal{L}} \leq 1.$$

On the other hand let $u_0 = \frac{1}{(t-s)^{1/2}} \in L^2(0,1)$. Then

$$\frac{\| (P_t - P_s)(u_0) \|_2^2}{\| u_0 \|_2^2} = \frac{\int_s^t u_0^2 dx}{\frac{1}{t-s}} = (t-s) \frac{1}{t-s} = 1.$$

$$\Rightarrow \| P_t - P_s \|_{\mathcal{L}} = 1$$

So, there is an uncountable set of operators such that the distance between any two of them is 1. So, $\mathcal{L}(H)$ is <u>not</u> separable.

QED

<u>Remark:</u> $B_{1/3}(P_t) = B_t \longrightarrow$ open $B_t \cap B_s = \emptyset$ if $t \neq s$.

This implies the nonseparability of $\mathcal{L}(H)$. To see this suppose that $\mathcal{L}(H)$ is separable. Then we can find $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(H)$ dense. For each $t \in (0,1)$ $B_t \cap \{A_n\}_{n\in\mathbb{N}} \neq \emptyset$ Choose $n_t \in \mathbb{N}$ s.t

$$A_{n_t} \in B_t$$

Then the map. $t \longrightarrow n_t$ is injective, because

$$n_t = n_s \Rightarrow A_{n_t} = A_{n_s} \in B_t \cap B_s \neq \emptyset \Rightarrow t = s.$$

$$\Rightarrow (0,1) \text{ is countable, contradiction}$$

<u>Exercise 2:</u> Let X be a Banach space, $A \in \mathcal{L}(X)$, $\|A\|_{\mathcal{L}} < 1$. Show that $I - A$ is invertible and $(I-A)^{-1} = \sum_{k\geq 0} A^k$

<u>Proof:</u> Note that $\sum_{k\geq 0} \|A^k\|_{\mathcal{L}} \leq \sum_{k\geq 0} \|A\|_{\mathcal{L}}^k = \dfrac{1}{1 - \|A\|_{\mathcal{L}}}$,

$$\Rightarrow \sum_{k\geq 0} A^k \text{ is absolutely convergent in } \mathcal{L}(X)$$

Also we have

$$(I - A) \sum_{k\geq 0} A^k = (I - A) + (A - A^2) + \dots = I$$

Similarly

$$\Big(\sum_{k\geq 0} A^k\Big)(I - A) = I$$

Therefore we conclude that $I - A$ is invertible

<u>QED</u>

Exercise 3: Let X be a Banach space and $\mathcal{U} \in \mathcal{L}(X)$ the set of invertible operators is open.

Proof: From Exercise 2, we have that

$\| I - A \|_{\mathcal{L}} < 1 \Rightarrow A$ is invertible and $A^{-1} = \sum_{k \geq 0} (I-A)^k$. Hence

$$\| A^{-1} \|_{\mathcal{L}} \leq \sum_{k \geq 0} \| I - A \|_{\mathcal{L}}^k = \frac{1}{1 - \| I - A \|_{\mathcal{L}}}$$

Suppose that $A_0 \in \mathcal{L}(X)$ is invertible. Then

$$I - A A_0^{-1} = (A_0 - A) A_0^{-1} \quad \forall A \in \mathcal{L}(H).$$

and so if

$$\| A_0 - A \|_{\mathcal{L}} < \frac{1}{\| A_0^{-1} \|_{\mathcal{L}}}$$

$$\Rightarrow \| I - A A_0^{-1} \|_{\mathcal{L}} < 1.$$

Hence if $\| A_0 - A \|_{\mathcal{L}} < \frac{1}{\| A_0^{-1} \|_{\mathcal{L}}}$, then A is invertible

(since $A A_0^{-1}$ is). Also

$$\| A^{-1} \|_{\mathcal{L}} = \| ((A A_0^{-1}) A_0)^{-1} \|_{\mathcal{L}} \leq \| A_0^{-1} \|_{\mathcal{L}} \, \| A_0 A^{-1} \|_{\mathcal{L}}$$

$$\leq \frac{\| A_0^{-1} \|_{\mathcal{L}}}{1 - \| A_0 - A \|_{\mathcal{L}} \| A_0^{-1} \|_{\mathcal{L}}}$$

QED

Remark: $A \longrightarrow A^{-1}$ is a homeomorphism of $\mathcal{U}$ onto $\mathcal{U}$

Exercise 4: Let X be a Banach space and $A, T \in \mathcal{L}(X)$ such that $AT = TA$. Show that

$$AT \text{ is invertible} \iff A \text{ and } T \text{ both invertible.}$$

Proof: $\Leftarrow$ We know that $(AT)^{-1} = T^{-1}A^{-1}$

$\Rightarrow$ First we show that both A and T are injective

If for $x \neq 0$, we have $T(x) = 0$, then $(AT)(x) = 0$ and so $(AT)(\cdot)$ is not injective.

If for $x \neq 0$, we have $A(x) = 0$, then since $AT = TA$, as above we infer that $(AT)(\cdot) = (TA)(\cdot)$ is not injective.

If $A(X) \neq X$, then $(AT)(\cdot)$ is not surjective

If $T(X) \neq X$, then $(TA)(\cdot) = (AT)(\cdot)$ is not surjective.

Therefore both $A(\cdot)$ and $T(\cdot)$ are 1-1, onto, thus invertible.

QED

Exercise 5 Let X be an infinite dimensional Banach space. Show that X_w is not metrizable

Proof: We know that $\overline{B}_1 = \overline{\partial B_1}^{w}$. If X_w is metrizable, then let $d(\cdot,\cdot)$ be the metric generating the weak topology. Since $0 \in \overline{\partial B_1}^{w}$

we can find $u_n \in \partial(n\bar{B}_1)$ such that

$$d(u_n, 0) < \frac{1}{n},$$

$$\Rightarrow u_n \xrightarrow{w} 0,$$

$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq X \text{ is bounded} \Rightarrow\Leftarrow.$$

QED

<u>Exercise 6:</u> Let X be a Banach space, $C \subseteq X$ compact and $\{u_n\}_{n\in\mathbb{N}} \subseteq C$ such that $u_n \xrightarrow{w} u$. Show that $u_n \to u$.

<u>Proof:</u> We can find a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ s.t

$$u_{n_k} \longrightarrow \hat{u} \text{ in } X$$

$$\Rightarrow \hat{u} = u.$$

Hence every subsequence of $\{u_n\}_{n\in\mathbb{N}}$ has a further subsequence which converges in norm to u. By the Urysohn criterion, for the original sequence we have

$$u_n \longrightarrow u \text{ in } X.$$

QED

<u>Exercise 7:</u> Let X be a Banach space $C \subseteq X$ w-closed, $K \subseteq X$ w-compact. Show that $C + K \subseteq X$ is w-closed.

<u>Proof:</u> Let $\{u_n\}_{n\in\mathbb{N}} \subseteq C+K$ and assume that $u_n \xrightarrow{w} u$. We have

$$u_n = c_n + k_n \qquad c_n \in C,\ k_n \in K.$$

By the Eberlein-Smulian theorem, we may assume that

$$k_n \xrightarrow{w} k \in K.$$

Then $u_n - k_n = c_n \xrightarrow{w} u - k$ in X and since $C =$ w-closed

$$c = u - k \in C,$$

$$\Rightarrow u = c + k \in C + K,$$

$$\Rightarrow C + K \text{ is w-closed}$$

QED

Exercise 8: Let X be a Banach space such that $\forall\, C \subseteq X$ closed convex

$$d(x, C) = \inf \left[\|x - c\| : c \in C \right]$$

is realized. Show that X is reflexive.

Proof: Arguing by contradiction, suppose that X is not reflexive. So, we can find $x^* \in \partial B_1^*$ such that

$$\langle x^*, x \rangle < \|x^*\|_* = 1 \qquad \forall x \in \overline{B}_1$$

Therefore

$$(x^*)^{-1}(1) \cap \overline{B}_1 = \emptyset,$$

$$\Rightarrow \|x\| > 1 \quad \forall x \in (x^*)^{-1}(1),$$

$$\Rightarrow d(0, \underbrace{(x^*)^{-1}(1)}_{\text{closed, convex}}) = 1$$

, a contradiction

QED

Exercise 9: Let X be a reflexive Banach space, $C \subseteq X$ closed, convex $u \in X \setminus C$. Show that there exists $c_0 \in C$ s.t

$$\| u - c_0 \| = d(u, C).$$

Proof: Let $\{c_n\}_{n\in\mathbb{N}} \subseteq C$ such that

$$\| u - c_n \| \downarrow d(u, C).$$

Then $\{c_n\}_{n\in\mathbb{N}} \subseteq C$ is bounded. By reflexivity we may assume that

$$c_n \xrightarrow{w} c_0 \in C \quad (\text{since } C \text{ is w-closed}),$$

$$\Rightarrow \| u - c_0 \| \leq \liminf_{n\to\infty} \| u - c_n \| = d(u, C)$$

$$\Rightarrow \| u - c_0 \| = d(u, C).$$

QED

Exercise 10: Let $\{u_n\}_{n\in\mathbb{N}} \subseteq L^p(\Omega)$ $(1<p<\infty)$, $u_n \xrightarrow{w} u$ and

$$\limsup \| u_n \|_p \leq \| u \|_p.$$

Show that $u_n \longrightarrow u$ in $L^p(\Omega)$

Proof: Since $u_n \xrightarrow{w} u$ in $L^p(\Omega)$ we have

$$\| u \|_p \leq \liminf_{n\to\infty} \| u_n \|_p,$$

$$\Rightarrow \| u_n \|_p \longrightarrow \| u \|_p.$$

But $L^p(\Omega)$ is uniformly convex. So, by the Kadec-Klee property, we have $u_n \longrightarrow u$ in $L^p(\Omega)$.

QED

Exercise 11: Let $V \subseteq \ell^1$ be a closed, infinite dimensional subspace. Show that V^* is <u>not</u> separable.

Proof: We proceed by contradiction.

So, suppose that V^* is separable. Then

$$(\overline{B}_1, w) \text{ is metrizable.}$$

Also we know that

$$0 \in \overline{\partial B_1}^{w}.$$

So, we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq \partial B_1$ such that

$$u_n \xrightarrow{w} 0 \quad \text{in } V,$$

$$\Rightarrow u_n \xrightarrow{w} 0 \quad \text{in } \ell^1$$

$$\Rightarrow u_n \longrightarrow 0 \quad \text{in } \ell^1 \text{ (Schur property)}$$

a contradiction since $\|u_n\| = 1 \quad \forall n \in \mathbb{N}$.

QED

Exercise 12: Let $S: \ell^2 \to \ell^2$ be defined by

$$S(u_1, u_2, \dots) = (0, u_1, \dots, u_n, \dots) \quad \forall \{u_n\}_{n \in \mathbb{N}} \subseteq \ell^2$$

(right or forward shift)

Show that S is <u>not</u> compact

Proof: $\{e_n\}_{n \in \mathbb{N}}$ standard o.n basis of ℓ^2. Then

$$\|e_n\| = 1 \quad \forall n \in \mathbb{N}, \quad \|S(e_n) - S(e_m)\| = \sqrt{2} \quad n \neq m$$

So $\{S(e_n)\}_{n\in\mathbb{N}}$ has no convergent subsequence

$\Rightarrow S \notin \mathcal{L}_c(\ell^2)$.

QED

Exercise ·13: Let X, Y be Banach spaces, $A \in \mathcal{L}_c(X,Y)$ and $R(A) \subseteq Y$ is closed. Show that $A \in \mathcal{L}_f(X,Y)$ and if in addition $\dim\ker A < \infty$, show that X is finite dimensional

Proof: Let $V = R(A) \subseteq Y$. Then V is a Banach space. The opera for $A: X \longrightarrow V$ is surjective. So, by the Open Mapping Thm

$\exists\ \delta > 0$ s.t

$$\delta B_1^V \subseteq A(B_1^X)$$

But since $A \in \mathcal{L}_c(X,Y) \Rightarrow A \in \mathcal{L}_c(X,V)$ and so

$$\overline{A(B_1^X)} \subseteq V \text{ is compact},$$

$$\Rightarrow \dim V < \infty,$$

$$\Rightarrow A \in \mathcal{L}_f(X,Y).$$

Let Z be the topological complement of $N(A) = \ker A$

$$X = N(A) \oplus Z$$

Let $\hat{A} = A\big|_Z$. Then $\hat{A}$ is bijective from Z onto V

Hence

$$Z \text{ and } V \text{ isomorphic}$$

$$\Rightarrow \dim V = \dim Z < \infty$$
$$\Rightarrow \dim X < \infty$$

QED

Exercise 14: Let X be a Banach space, $Y \subseteq X$ a subspace and $P: X \longrightarrow Y$ a projection. Show that Y is closed

Proof: Let $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ and assume that $y_n \longrightarrow y$ in Y. Then

$$P(y_n) \longrightarrow P(y)$$

Since $P(y_n) = y_n \ \forall n \geqslant 1$ we also have

$$y_n \longrightarrow P(y),$$

$$\Rightarrow y = P(y),$$

$$\Rightarrow y \in Y.$$

QED

Exercise 15: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces with X reflexive and $A \in \mathcal{L}_c(X,Y)$. Let $|\cdot|_X$ be another norm on X weaker than $\|\cdot\|_X$. Show that $\forall \varepsilon > 0$ we can find $c_\varepsilon > 0$ such that

$$\|A(u)\|_Y \leqslant \varepsilon \|u\|_X + c_\varepsilon |u|_X \quad \forall u \in X$$

Proof: Arguing indirectly, suppose we can find $\varepsilon > 0$ and $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\|u_n\|_X = 1 \qquad \|A(u_n)\|_Y > \varepsilon + n|u_n|_X \tag{*}$$

Since $(X, \|\cdot\|_X)$ is reflexive, $\overline{B}_1^X$ = w-compact and so by the Eberlein-Smulian theorem we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } (X, \|\cdot\|_X)$$

Since $|\cdot|_X$ is weaker than $\|\cdot\|_X$, we have that

$$id : (X, \|\cdot\|_X) \longrightarrow (X, |\cdot|_X)$$

is continuous, linear, hence w-continuous. Therefore

$$u_n \xrightarrow{w} u \quad \text{in } (X, |\cdot|_X)$$

Also since $A \in \mathcal{L}_c(X,Y)$, we have

$$A(u_n) \longrightarrow A(u) \quad \text{in } Y$$

Then from $(*)$ we infer that

$$|u_n|_X \longrightarrow 0,$$

$$\Rightarrow u = 0 \quad \text{and so } A(u) = 0$$

On the other hand from $(*)$

$$\|A(u_n)\|_Y \geq \varepsilon \quad \forall n \in \mathbb{N},$$

$$\Rightarrow \|A(u)\|_Y \geq \varepsilon \text{, a contradiction}$$

QED

Dfn: H = Hilbert space over $\mathbb{C}$ and $A : H \longrightarrow H$ linear

Numerical Range of A is the set

$$W(A) = \{ (A(u), u) : \|u\| = 1 \}$$

When this set is bounded, then

$$w(A) = \sup \{ |(A(u), u)| : \|u\| = 1 \}$$

is the numerical radius.

Remarks: If $A \in \mathcal{L}(H)$, then

$$|(A(u), u)| \leq \|A\|_{\mathcal{L}} \quad \forall \|u\| = 1$$

Hence $W(A) \subseteq \mathbb{C}$ is bounded and

$$w(A) \leq \|A\|_{\mathcal{L}}$$

Also $A = 0$ if and only if $W(A) = \{0\}$. This is not true for H real Hilbert space. Consider $(\mathbb{R}^2, \|\cdot\|_2)$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For any $u = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ we have

$$(A(u), u) = (t_2, -t_1) \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = 0 \quad \text{but } A \not\equiv 0.$$

Exercise 16: Let H be a Hilbert space, $A \in \mathcal{L}(H)$ and $\lambda \notin \overline{W(A)}$. Show that $A - \lambda I$ is an isomorphism

Proof: We have $c = d(\lambda, \overline{W(A)}) > 0$

For $u \in H$, $\|u\| = 1$, we have

$$\|(A - \lambda I)(u)\| \geq |((A - \lambda I)(u), u)| = |(A(u), u) - \lambda| \geq c, \quad (*)$$

$$\Rightarrow \|(A - \lambda I)(u)\| \geq c\|u\| \quad \forall u \in H.$$

$$\Rightarrow R(A - \lambda I) \subseteq H \text{ closed}$$

Suppose that $R(A-\lambda I)$ is a proper subspace of H. So, we can find $\bar{u} \in H$, $\|\bar{u}\| = 1$ s.t

$$\bar{u} \perp R(A-\lambda I),$$

$$\Rightarrow \left((A-\lambda I)(\bar{u}), \bar{u} \right) = 0$$

which contradicts $(*)$.

Therefore $R(A-\lambda I) = H$ and so $A-\lambda I$ is an isomorphism

QED

Exercise 17: Let H a Hilbert space and $A \in \mathcal{L}(H)$ unitary.
Show that $\forall \lambda \in \mathbb{C}$, $|\lambda| \neq 1$, $A-\lambda I =$ isomorphism

Proof: Since $A =$ unitary, it is an isometry and so

$$\|A\|_{\mathcal{L}} = 1$$

Suppose $\lambda \in \mathbb{C}$, $|\lambda| > 1$. Then $\lambda \in \rho(A)$ and so

$$A-\lambda I = \text{isomorphism}$$

Now suppose $|\lambda| < 1$. We know $A^* =$ unitary too

$$A^* - \frac{1}{\lambda} I = \text{isomorphism}$$

But we have

$$A-\lambda I = -\lambda \left(A^* - \frac{1}{\lambda} I \right) A,$$

$$\Rightarrow A-\lambda I \quad \text{isomorphism}$$

QED

Exercise 18: Let H be a Hilbert space and P, Q orthogonal projections. Show that

$$PQ = \text{orthogonal projection } \underline{\text{iff}} \ PQ = QP.$$

Moreover, $R(PQ) = R(P) \cap R(Q)$.

Proof: $\Rightarrow$ Since PQ is orthogonal projection is s.a So

$$\begin{aligned}(PQ(u), v) &= (u, PQ(v)) \\ &= (P(u), Q(v)) \\ &= (QP(u), v) \quad \forall u, v \in H\end{aligned}$$

$$\Rightarrow PQ = QP.$$

$\Leftarrow$ We have $PQ = QP$. Then

$$(PQ)^2 = PQPQ = P^2Q^2 = PQ$$

Also.

$$\begin{aligned}(PQ(u), v) &= (Q(u), P(v)) \\ &= (u, QP(v)) \\ &= (u, PQ(v)) \quad \forall u, v \in H\end{aligned}$$

$\Rightarrow PQ$ is s.a

Therefore PQ is an orthogonal projection.

Let $u \in R(P) \cap R(Q)$. Then

$$u = P(u) \quad , \quad u = Q(u).$$

$$\Rightarrow u = PQ(u)$$

$$\Rightarrow R(P)\cap R(Q) \subseteq R(PQ).$$

Now let $u \in R(PQ)$. Then

$$u = PQ(u) = P(Q(u)) \quad \text{and so} \quad u \in R(P).$$

But also $u \in R(QP)$ and so $u \in R(Q)$. Hence

$$u \in R(P)\cap R(Q),$$

$$\Rightarrow R(PQ) = R(P)\cap R(Q).$$

QED

Exercise 19: Let H be a Hilbert space, $A \in \mathcal{L}(H)$ and $V \subseteq H$ closed subspace. Show that

$$V \text{ is } A\text{-invariant } \underline{\text{iff}} \ V^{\perp} \text{ is } A^{*}\text{-invariant}$$

Proof: $\Rightarrow$ V is A-invariant and so $A(V) \subseteq V$. Then if $y \in V^{\perp}$ we have

$$(A(u), y) = 0 \quad \forall u \in V,$$

$$\Rightarrow (u, A^{*}(y)) = 0 \quad \forall u \in V$$

$$\Rightarrow A^{*}(y) \in V^{\perp},$$

$$\Rightarrow A^{*}(V^{\perp}) \subseteq V^{\perp}$$

$\Leftarrow$ This is proved similarly.

QED

Exercise 20: Let $H = \ell^2$ and consider the operator

$$A(\{\lambda_n\}_{n\in\mathbb{N}}) = \{\vartheta_n \lambda_n\}_{n\in\mathbb{N}}$$

with $\vartheta_n \to 0$. Show that $A \in \mathcal{L}_c(\ell^2)$.

Proof: For each $n \in \mathbb{N}$ consider the finite rank operator

$$F_m(\{\lambda_n\}_{n\in\mathbb{N}}) = (\vartheta_1\lambda_1, \ldots, \vartheta_m\lambda_n, 0, \ldots 0, \ldots)$$

Clearly F_m is linear continuous.

Since $\vartheta_n \to 0$, given $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ s.t

$$|\vartheta_n| \leq \varepsilon \quad \forall n \geq n_0$$

We have for $u = \{\lambda_n\}_{n\in\mathbb{N}}$, and $m \geq n_0$

$$\|(A - F_m)(u)\| = \Big(\sum_{k\geq m+1} \vartheta_k^2 \lambda_k^2\Big)^{1/2} \leq \varepsilon \Big(\sum_{k\geq m+1} \lambda_k^2\Big)^{1/2}$$

$$\leq \varepsilon \|u\|_{\ell^2}$$

$$\Rightarrow \|A - F_m\|_{\mathcal{L}} \leq \varepsilon \quad \forall m \geq n_0$$

$$\Rightarrow A \in \mathcal{L}_c(\ell^2).$$

QED

Exercise 21: Let $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^p(\Omega)$ $1 < p < \infty$ and assume that

$$\|u_n\|_p \longrightarrow \|u\|_p$$

$$u_n \xrightarrow{a.e} u$$

Show that $u_n \longrightarrow u$ in $L^p(\Omega)$.

Proof: Evidently $\{u_n\}_{n\in\mathbb{N}} \subseteq L^p(\Omega)$ is bounded. Since $L^p(\Omega)$ is reflexive, we can find a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ s.t

$$u_{n_k} \xrightarrow{w} \hat{u} \quad \text{in } L^p(\Omega),$$

$$\Rightarrow \int_A u_{n_k}\,dx \longrightarrow \int_A \hat{u}\,dx \quad \forall A \subseteq \Omega \text{ measurable}$$

Also $\{\chi_A u_n\}_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ is uniformly integrable and

$$\chi_A u_n \xrightarrow{a.e} \chi_A u$$

So, by Vitali's theorem

$$\int_A u_n\,dx = \int_\Omega \chi_A u_n\,dx \longrightarrow \int_\Omega \chi_A u\,dx = \int_A u\,dx$$

$$\Rightarrow \int_A \hat{u}\,dx = \int_A u\,dx \quad \forall A \subseteq \Omega \text{ measurable},$$

$$\Rightarrow \hat{u} = u$$

So $u_{n_k} \xrightarrow{w} u$ in $L^p(\Omega)$

$$\|u_{n_k}\|_p \longrightarrow \|u\|_p.$$

$$\Downarrow \text{ Kadec-Klee}$$

$$u_{n_k} \longrightarrow u \text{ in } L^p(\Omega),$$

$$\Rightarrow u_n \longrightarrow u \text{ in } L^p(\Omega)$$

(by the Urysohn criterion).

QED

An alternative proof that covers also the case $p=1$

Brezis-Lieb Lemma

If $1 \le p < \infty$, $\{u_n\}_{n\in\mathbb{N}} \subseteq L^p(\Omega)$ bounded and

$$u_n \xrightarrow{a.e} u$$

then $\lim\limits_{n\to\infty} \left[\|u_n\|_p^p - \|u_n - u\|_p^p \right] = \|u\|_p^p$

Exercise 22: Let X be a closed infinite dimensional subspace of ℓ^1. Show that X^* is nonseparable

Proof We know that $0 \in \overline{\partial B_1^X}^w$. If X^* is separable, then

$$(\overline{B}_1^X, w) = \text{metrizable}$$

So, we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \partial B_1^X$ s.t

$$u_n \xrightarrow{w} 0 \quad \text{in } X,$$

$$\Rightarrow u_n \xrightarrow{w} 0 \quad \text{in } \ell^1,$$

$$\Rightarrow u_n \to 0 \text{ in } \ell^1 \text{ (by Schur property)}$$

a contradiction.

QED

Exercise 23: Show that a separable Banach space X with a nonseparable dual is not reflexive.

Proof: Suppose that X is reflexive. Then the canonical embedding $\hat{i}: X \to X^{**}$ is an isometric isomorphism. Hence

X^{**} = separable,

$\Rightarrow X^*$ = separable, a contradiction.

QED

Exercise 24: Let X, Y be infinite dimensional Banach spaces $A \in \mathcal{L}(X,Y)$ and $\|A(u)\| \geq c\|u\|$ $\forall u \in X$ with $c>0$. Is A compact? Explain

Proof: NO Let $V = A(X)$. We know $V \subseteq Y$ is closed and

$A: X \to V$ is a bijection,

$\Rightarrow A \in \mathcal{L}(X,V)$ is an isomorphism (Banach Thm),

$\Rightarrow A^{-1}(A(\bar{B}_1^X)) = \bar{B}_1^X$ = compact

$\Rightarrow X$ = finite dimensional, a contradiction.

QED