EXERCISES 1

Theorem (Grothendieck).

If X is a Banach space and $A \subseteq X$ then A is relatively compact iff $\exists \{u_n\}_{n\in\mathbb{N}} \subseteq X$ $u_n \to 0$ s.t

$$A \subseteq \overline{\operatorname{conv}}\{u_n\}_{n\in\mathbb{N}}.$$

Square Root of an Operator.

Let H be a Hilbert space and $A \in \mathcal{L}(H)$ s.a and $A \geqslant 0$ (that is, $(A(u),u) \geqslant 0 \ \forall u \in H$)

Then $\exists!\ T \in \mathcal{L}(H)$, T s.a, $T \geqslant 0$ s.t

$$T^2 = A$$

We write $T = \sqrt{A}$.

Remark: T commutes with any operator which commutes with A.

Exercise 1 Let $X = C[0,1]$ and let $A: X \to X$ be defined by

$$A(u)(x) = \int_0^x u(s)\,ds \quad \forall u \in X$$

Show that $A \in \mathcal{L}_c(X)$ and find its norm

Proof: Clearly $A(\cdot)$ is a linear operator.

Also $\|A(u)\| = \sup\limits_{x\in[0,1]} \left|\int_0^x u(s)\,ds\right| \leqslant \|u\| \quad \forall u\in X$

$$\Rightarrow A \in \mathcal{L}(X) \quad \text{and} \quad \|A\|_{\mathcal{L}} \leqslant 1.$$

Let $D \subseteq X$ bounded and let $u \in A(D)$. Then for $t_1, t_2 \in [0,1]$ $t_1 < t_2$, we have

$$|u(t_2) - u(t_1)| = \left|\int_{t_1}^{t_2} v(s)\,ds\right| \qquad v \in D$$

$$\leqslant \int_{t_1}^{t_2} |v(s)|\,ds \leqslant M(t_2 - t_1)$$

$\Rightarrow A(D) \subseteq X$ equicontinuous.

Also

$$\|u\| = \sup_{0\leqslant x\leqslant 1} \left|\int_0^x v(s)\,ds\right| \leqslant \sup_{0\leqslant x\leqslant 1} \int_0^x |v(s)|\,ds \leqslant \|v\| \leqslant M.$$

$\Rightarrow A(D)$ is bounded

By the Arzela-Ascoli theorem we conclude that $\overline{A(D)}^{\|\cdot\|} \subseteq X$ is compact,

$$\Rightarrow A \in \mathcal{L}_c(X)$$

Consider the constant function $\bar{v}(x) = 1 \quad \forall x \in [0,1]$

Then $\|\bar{v}\| = 1$. We have

$$\|A(\bar{v})\| = \sup_{0\leqslant x\leqslant 1} \left|\int_0^x ds\right| = \sup_{0\leqslant x\leqslant 1} |x| = 1$$

$$\Rightarrow \|A\|_{\mathcal{L}} = 1$$

QED

<u>Exercise 2:</u> Let X be a Banach space, $A \in \mathcal{L}_c(X)$ and assume that $A^2 = A$. Show that $A \in \mathcal{L}_f(X)$.

Proof: Let $V = A(X)$ (the range of $A(\cdot)$)

Let $\hat{A} = A\big|_V$. If $v \in V$, then $\exists\, x \in X$ such that

$$A(x) = v,$$

$$\Rightarrow A(A(x)) = A(v),$$

$$\Rightarrow A(x) = A(v),$$

$$\Rightarrow v = A(v).$$

So we have proved that $\hat{A} = id$. Since $A \in \mathcal{L}_c(X)$ it follows that $\hat{A} = id \in \mathcal{L}_c(V)$ and so $\dim V < \infty$. Therefore $A \in \mathcal{L}_f(X)$.

QED

<u>Exercise 3:</u> Let X, Y, V be Banach spaces, $B: X \times Y \to V$ a bilinear operator which is separately continuous. Show that $B(\cdot,\cdot)$ is continuous.

<u>Proof:</u> $\forall x \in X$ $B_x: Y \to V$ is defined by $B_x(y) = B(x,y)$. Hence

$$B_x \in \mathcal{L}(Y, V)$$

Consider

$$C = \{ v^* \circ B_x : \|v^*\|_{V^*} \leq 1, \|x\|_X \leq 1 \} \subseteq Y^*$$

Let $y \in Y$. Then

$$\{ (v^* \circ B_x)(y) : \|v^*\|_{V^*} \leq 1, \|x\|_X \leq 1 \} \subseteq \mathbb{R} \text{ is bounded}$$

$$\Big[\ |(v^* \circ B_x)(y)| = |v^*(B(x,y))| = |(v^* \circ B_y)(x)|$$
$$\leq \|v^*\|_{V^*} \|B_y\|_{\mathcal{L}(X,V)} \|x\|$$
$$\leq \|B_y\|_{\mathcal{L}(X,V)} \ \Big]$$

$\Rightarrow$ $C \subseteq Y^*$ is w^*-bounded.

By the Uniform Boundedness Principle we have

$$\sup \Big[\|v^* \circ B_x\| : \|v^*\|_{V^*} \leq 1, \|x\|_X \leq 1 \Big] = M < \infty$$

$$\Rightarrow |v^* \circ B(x,y)| \leq M \|y\|_Y \quad \forall \|v^*\|_{V^*} \leq 1 \ \forall \|x\|_X \leq 1,$$

$$\Rightarrow |B(x,y)| \leq M \|y\|_Y \quad \forall \|x\| \leq 1,$$

$$\Rightarrow |B(x,y)| \leq M \|x\|_X \|y\|_Y,$$

$\Rightarrow$ $B(\cdot,\cdot)$ is continuous.

QED

Exercise 4: Let H be a Hilbert space, $A \in \mathcal{L}(H)$, s.a, $A \geq 0$. Show that the following statements are equivalent.

(a) $A(H)$ is dense in H.

(b) $N(A) = \{0\}$

(c) $A(\cdot)$ is positive definite (i.e., $(A(u),u) > 0 \ \forall u \neq 0$)

Proof: (a) $\Rightarrow$ (b) : Let $A(u)=0$ So, for all $h \in H$ we have

$$(A(u),h)=0,$$
$$\Rightarrow (u, A(h)) = 0,$$
$$\Rightarrow u \perp R(A),$$
$$\Rightarrow u = 0.$$

Therefore $N(A)=\{0\}$

(b) $\Rightarrow$ (c) ; Let $T=\sqrt{A} \in \mathcal{L}(H)$. Suppose

$$(A(x),x)=0,$$
$$\Rightarrow (T^2(x),x)=0,$$
$$\Rightarrow \|T(x)\|=0,$$
$$\Rightarrow T(x)=0,$$
$$\Rightarrow A(x)=T(T(x))=0, \text{ and so } x=0.$$

(c) $\Rightarrow$ (a) Suppose $A(H) \subseteq H$ is not dense Then we can find $u \in H, u \neq 0$ s.t

$$u \perp A(H),$$
$$\Rightarrow (u, A(u)) = (A(u), u) = 0,$$
$$\Rightarrow u = 0, \text{ a contradiction}$$

QED

Exercise 5: Let H be a Hilbert space, $A \in \mathcal{L}(H)$, A is s.a and $A \geq 0$. Show that A is bijective iff $\exists\, \epsilon > 0$ such that $A - \epsilon I$ is positive definite.

<u>Proof:</u> $\Rightarrow$ Let $T = \sqrt{A}$. Then T is bijective. By Banach's Theorem

$T^{-1} \in \mathcal{L}(H)$ and so

$$\|T^{-1}(h)\| \leq \vartheta \|h\| \quad \forall h \in H,$$

$$\Rightarrow \|u\| \leq \vartheta \|T(u)\| \quad \forall u \in H,$$

$$\Rightarrow \|u\|^2 < 2\vartheta^2 \|T(u)\|^2 \quad \forall u \in H \setminus \{0\}$$

$$\Rightarrow c_0 \|u\|^2 < (T^2(u), u)$$

for all $u \in H \setminus \{0\}$, with $c_0 = \frac{1}{2\theta^2}$

$$\Rightarrow ((A - c_0 I)(u), u) > 0 \quad \forall u \in H \setminus \{0\}.$$

$$\Rightarrow T - c_0 I \text{ is positive definite.}$$

$\Leftarrow$ Suppose $A - \varepsilon I$ positive definite. Then

$$\varepsilon \|u\|^2 \leq (A(u), u) \quad \forall u \in H,$$

$$\Rightarrow \varepsilon \|u\|^2 \leq \|T(u)\|^2 \quad \forall u \in H,$$

$$\Rightarrow \sqrt{\varepsilon} \|u\| \leq \|T(u)\| \quad \forall u \in H.$$

It follows that $T(\cdot)$ is injective and $R(T)$ is closed. Let $u \in R(T)^{\perp}$. Then

$$(u, T(h)) = 0 \quad \forall h \in H,$$

$$\Rightarrow (T(u), h) = 0 \quad \forall h \in H,$$

$$\Rightarrow T(u) = 0 \text{ and so } u = 0,$$

$$\Rightarrow T(\cdot) \text{ is surjective.}$$

$\Rightarrow T(\cdot)$ is bijective,

$\Rightarrow A = T^2$ is bijective.

QED

Exercise 6: Let H be a Hilbert space and $A \in \mathcal{L}(H)$ is s.a. Suppose that $(A(u), u) = 0 \ \forall u \in H$. Show that $A = 0$

Proof: Since A is s.a we know (Prop. 3.6.16)

$$\|A\|_{\mathcal{L}} = \sup\left[\, |(A(u), u)| : \|u\| \leq 1 \right]$$

$$\Rightarrow \|A\|_{\mathcal{L}} = 0 \quad \text{and so} \quad A = 0.$$

QED

Exercise 7: Let H be a Hilbert space, $A \in \mathcal{L}(H)$ is s.a and $A \geq 0$. Show that

A = compact iff $\sqrt{A}$ is compact

Proof: $\Rightarrow$ Let $\{u_n\}_{n \in \mathbb{N}} \subseteq \overline{B}_1$. Since A is compact we can find $\{u_{n_k}\}_{k \in \mathbb{N}}$ subsequence of $\{u_n\}_{n \in \mathbb{N}}$ such that

$$A(u_{n_k}) \longrightarrow h \text{ in } H.$$

For every $x \in H$, we have

$$\|\sqrt{A}(x)\|^2 = (\sqrt{A}(x), \sqrt{A}(x)) = (A(x), x) \leq \|A(x)\| \cdot \|x\|$$

(since $\sqrt{A}$ is s.a)

Therefore

$$\| \sqrt{A}(u_{n_k}) - \sqrt{A}(u_{n_\ell}) \| \leq \| A(u_{n_k} - u_{n_\ell}) \| \, \| u_{n_k} - u_{n_\ell} \|$$

$$\leq 2 \| A(u_{n_k} - u_{n_\ell}) \|,$$

$\Rightarrow \{ \sqrt{A}(u_{n_k}) \}_{k \in \mathbb{N}} \subseteq H$ Cauchy, thus convergent

$\Rightarrow \sqrt{A}$ is compact

$\underline{\Leftarrow}$ Follows from the ideal property of compact operators (since $A = \sqrt{A}\sqrt{A}$).

QED

Exercise 8: Let H be a Hilbert space, $A \in \mathcal{L}(H)$ and $\|A\|_{\mathcal{L}} \leq 1$.

Show that (a) $I - A^*A \geq 0$;

(b) if $T = (I - A^*A)^{1/2}$

then $\|u\|^2 = \|A(u)\|^2 + \|T(u)\|^2$ $\forall u \in H$;

(c) $\{ u \in H : \|u\| = \|A(u)\| \} = N(T)$

Proof: (a) $(I - A^*A)^* = I - A^*A \Rightarrow I - A^*A$ s.a

Also for any $u \in H$ we have

$$\big((I - A^*A)(u), u \big) = \|u\|^2 - \|A(u)\|^2 \geq 0.$$

$\Rightarrow I - A^*A \geq 0 \Rightarrow (I - A^*A)^{1/2} \geq 0$ exists.

(b) We have

$$\|T(u)\|^2 = (T(u),T(u)) = (T^2(u),u)$$
$$= ((I-A^*A)(u),u)$$
$$= \|u\|^2 - \|A(u)\|^2,$$

$$\Rightarrow \|u\|^2 = \|A(u)\|^2 + \|T(u)\|^2 \quad \forall u \in H$$

(c) Follows from (b)

QED

Exercise 9: Let H be a Hilbert space, $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(H)$ and $\lim_{n\to\infty} (A_n(u),h)=0 \quad \forall u,h \in H$. Is it true that

$$\|A_n\|_{\mathcal{L}} \longrightarrow 0 ?$$

Also can we say that $\{\|A_n\|_{\mathcal{L}}\}_{n\in\mathbb{N}}$ is bounded

Proof: For the first question the answer is negative

To see this, let $H=\ell^2$ and let

$$A_n(\{u_k\}_{k\in\mathbb{N}}) = (\overbrace{0,\dots,0}^{n\text{-entries}}, u_1, u_2 \dots)$$

We see that $\|A_n\|_{\mathcal{L}} = 1 \quad \forall n \in \mathbb{N}$ (so $\|A_n\|_{\mathcal{L}} \not\to 0$) and

for $\hat{u} = (u_k)_{k\in\mathbb{N}}$, $\hat{v} = (v_k)_{k\in\mathbb{N}}$, we have

$$|(A_n(\hat{u}),\hat{v})| \le \left(\sum_{k=n+1}^{\infty} |u_{k-n}|^2\right)^{1/2} \left(\sum_{k=n+1}^{\infty} |v_k|^2\right)^{1/2}$$
$$= \|\hat{u}\| \left(\sum_{k=n+1}^{\infty} |v_k|^2\right)^{1/2} \longrightarrow 0 \quad \text{as } n\to\infty.$$

Next we show that $\{\|A_n\|_{\mathcal{L}}\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ is bounded

Let $u\in H$ and consider $\{A_n(u)\}_{n\in\mathbb{N}} \subseteq H$. Then for all $h\in H$ the real sequence $\{(A_n(u), h)\}_{n\in\mathbb{N}}$ is bounded (being convergent)

Then by the uniform boundedness principle we have

$$\{A_n(u)\}_{n\in\mathbb{N}} \subseteq H \text{ is bounded}$$

Again the uniform boundness principle implies that

$$\|A_n\|_{\mathcal{L}} \leq M \quad \forall\, n\in\mathbb{N}.$$

QED

Exercise 10: Let X be a Banach space and $K\subseteq X$ w-compact. Show that K is norm closed and norm bounded

Proof: Note that $w\leq s$ and boundedness is duality invariant.

QED