Online Learning and Online Convex Optimization

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(Projected) Gradient Descent of convex $f : S \to \mathbb{R}$ on convex $\subseteq \mathbb{R}^d$:

$$\boldsymbol{x}_{t+1} = \arg\min_{\boldsymbol{x}\in S} \left\| \left[\boldsymbol{x}_t - \eta \nabla f(\boldsymbol{x}_t)\right] - \boldsymbol{x} \right\|$$

Theorem : For step size $\eta_t = B/(G\sqrt{t})$ and #steps $T \geq B^2G^2/\varepsilon^2$,

$$f\Big(\sum_t \boldsymbol{x}_t/T\Big) \leq \Big(\sum_t f(\boldsymbol{x}_t)\Big)/T \leq f(\boldsymbol{x}^*) + \varepsilon$$

Convex Optimization – Projected Gradient Descent

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ML applications: Gradient Descent step on all training data is **too expensive** ; **online learning** through online convex optimization!

Online Learning Revisited

We fix $\mathcal{H}$ and loss ℓ (known to algo) [possibly also a $\mathcal{D}$ (unknown to algo)].

On each step $t = 1, \dots, T$:

1. Learner picks hypothesis $h_t \in \mathcal{H}$
2. Training example $(\boldsymbol{x}_t, y_t)$ is chosen (may be from $\mathcal{D}$, but it may be chosen by **adversary**)
3. Learner incurs loss $\ell_t(h_t, (\boldsymbol{x}_t, y_t))$

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Goal is to minimize **regret** :

$$\mathrm{Regret}(T) = \sup_{(\boldsymbol{x}_1, y_1), \dots (\boldsymbol{x}_T, y_T)} \left(\sum_{t=1}^{T} \ell_t(h_t, (\boldsymbol{x}_t, y_t)) - \min_{h^* \in \mathcal{H}} \sum_{t=1}^{T} \ell_t(h^*, (\boldsymbol{x}_t, y_t)) \right)$$

(Online) algorithm is **no-regret** if $\mathrm{Regret}(T)/T \to 0$ as $T \to \infty$

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Any no-regret online algorithm can be used for learning!

We focus on **regret minimization** for this and next lecture.

Online Learning: Basic Setting

Two actions: H and L (binary classification).

On each day $t = 1, \dots, T$:

1. Learner **picks action** $i_t \in \{H, L\}$
2. Adversary **picks loss** vector $\boldsymbol{\ell}_t = (\ell_t^H, \ell_t^L) \in [0,1]^2$
3. Learner learns $\boldsymbol{\ell}_t$ and **incurs loss** $\ell_t^{i_t}$

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Goal is to minimize **regret** (loss wrt. **best fixed** action in hindsight):

$$\mathrm{Regret}(T) = \sup_{\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_T} \left(\sum_{t=1}^{T} \ell_t^{i_t} - \min_{i \in \{H,T\}} \sum_{t=1}^{T} \ell_t^i \right)$$

(Online learning) algorithm is **no-regret** if $\mathrm{Regret}(T)/T \to 0$ as $T \to \infty$

Online Learning: Follow the Leader

Follow the Leader (FTL):

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1. **Deterministic** action choice, given the past (randomness always helps against the unknown).
2. Action choices can be very **unstable** (different choice each day).

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Proof : start with $(0, 1/2)$-loss; then loss for action i_t (chosen by the algorithm) $= 1$, and loss for other action $= 0$.

Any deterministic algorithm incurs loss $= T$, while best action incurs loss $\leq T/2$.

Online Learning: Randomization

Two actions: H and L (binary classification).

On each day $t = 1, \ldots, T$:

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2. Adversary picks loss vector $\boldsymbol{\ell}_t = (\ell_t^H, \ell_t^L) \in [0, 1]^2$
3. Learner learns $\boldsymbol{\ell}_t$ and incurs **expected loss**

$$f(p_t; \boldsymbol{\ell}_t) = p_t \ell_t^H + (1 - p_t)\ell_t^L$$

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$$f(p_t; \boldsymbol{\ell}_t) = p_t \ell_t^H + (1 - p_t)\ell_t^L$$

Goal is to minimize **expected regret** :

$$\text{Exp-Regret}(T) = \sup_{\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_T} \left(\sum_{t=1}^{T} f(p_t; \boldsymbol{\ell}_t) - \min_{p \in [0,1]} \sum_{t=1}^{T} f(p; \boldsymbol{\ell}_t) \right)$$

Randomization potentially allows for improved **stability** .

Online Learning: (Randomized) Follow the Leader

Follow the Leader (FTL):

$$p_t = \arg\min_{p \in [0,1]} \sum_{\tau=1}^{t-1} f(p; \ell_\tau) = \arg\min_{p \in [0,1]} F_{t-1}(p)$$

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Theorem : For any loss sequence $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$, FTL has:

$$\text{Exp-Regret}_{FTL}(T) = \underbrace{\sum_{t=1}^{T} f(p_t; \boldsymbol{\ell}_t) - \min_{p \in [0,1]} \sum_{t=1}^{T} f(p; \boldsymbol{\ell}_t)}_{\text{expected regret}} \leq \underbrace{\sum_{t=1}^{T} |p_t - p_{t+1}|}_{\text{instability}}$$

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For the analysis, we define **Be the Leader** (BTL):

$$p_t^* = \arg\min_{p \in [0,1]} \sum_{\tau=1}^{t} f(p; \ell_\tau) = \arg\min_{p \in [0,1]} F_t(p)$$

Regret of Be the Leader

Lemma: For any loss sequence $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$, $\text{Regret}_{BTL}(T) \leq 0$

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By **induction** on t, we show that for any $t \geq 1$:

$$\underbrace{\sum_{\tau=1}^{t} f(p_\tau^*; \boldsymbol{\ell}_t)}_{\text{loss of BTL up to } t} \leq \underbrace{\min_{p \in [0,1]} F_t(p)}_{\text{loss of best fixed action up to } t} = \underbrace{F_t(p_t^*)}_{\text{by definition of } p_t^*}$$

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$$\sum_{\tau=1}^{t+1} f(p_\tau^*; \boldsymbol{\ell}_\tau) = f(p_{t+1}^*; \boldsymbol{\ell}_{t+1}) + \sum_{\tau=1}^{t} f(p_\tau^*; \boldsymbol{\ell}_\tau)$$

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&\leq f(p_{t+1}^*; \boldsymbol{\ell}_{t+1}) + \min_{p \in [0,1]} F_t(p) \qquad \text{induction hypth.}
\end{aligned}$$

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&\le f(p^*_{t+1}; \boldsymbol{\ell}_{t+1}) + F_t(p^*_{t+1}) && F_t(p^*_t) \le F_t(p^*_{t+1}) \\
&= F_{t+1}(p^*_{t+1}) && \text{by dfn of } F_{t+1}(p)
\end{aligned}$$

Regret of FTL Against BTL

Lemma : For any loss sequence $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$,

$$\text{Regret}_{FTL}(T) \leq \text{Regret}_{BTL}(T) + \underbrace{\sum_{t=1}^{T} |p_t - p_{t+1}|}_{\text{instability}}$$

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&\leq \sum_{t=1}^{T} f(p_t^*; \boldsymbol{\ell}_t) + \sum_{t=1}^{T} |p_t - p_t^*| && \text{losses } \boldsymbol{\ell}_t \in [0,1]^2
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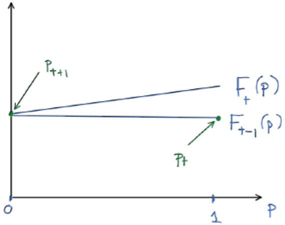
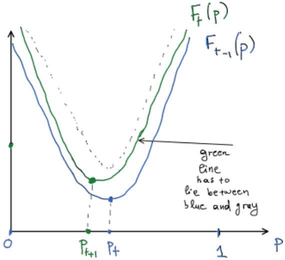
\end{aligned}$$

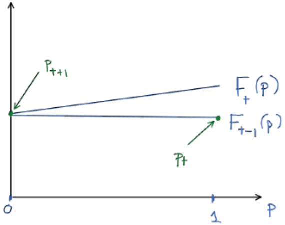
Convexity and Stability



(a) Two linear functions that are close to each other can have very far minima.



(b) For convex functions, closeness of the functions implies closeness of their minima.



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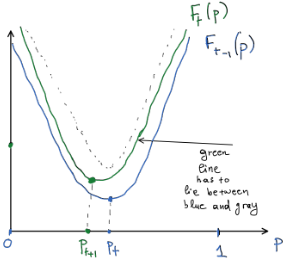

(b) For convex functions, closeness of the functions implies closeness of their minima.

$1/\eta$-**strongly convex** function $f : S \to \mathbb{R}$ wrt norm $\|\cdot\|$, if $\forall\, x, y \in S$:

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{1}{2\eta}\|x - y\|^2$$

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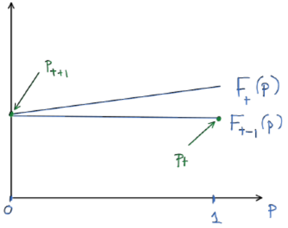

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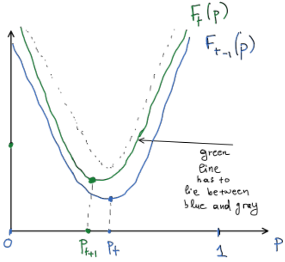

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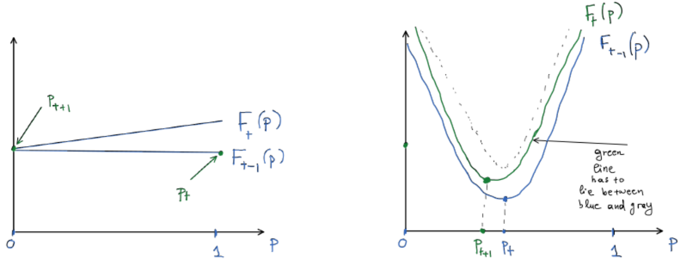
Functions $f, g : S \to \mathbb{R}$ be $1/\eta$-strongly convex wrt some norm $\|\cdot\|$ and $h(x) = g(x) - f(x)$ be L-Lipschitz wrt $\|\cdot\|$.



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Then, $\|x_f^* - x_g^*\| \leq \eta \cdot L$, with x_f^*, x_g^* minimizers of f, g.

Convexity and Stability

Functions $f, g : [0,1] \to \mathbb{R}$ be $1/\eta$-strongly convex and $h(x) = g(x) - f(x)$ be L-Lipschitz.
Then, $|p_f - p_g| \leq \eta \cdot L$, with p_f, p_g minimizers of f, g.

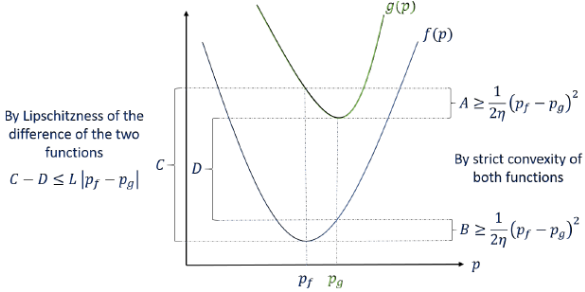

Figure 3: The proof of Lemma 3 follows immediately by noting that $C - D = A + B$ in the above figure, together with the fact that $C - D \leq L|p_f - p_g|$ by Lipschitzness of the difference of the two functions and $A + B \geq \frac{1}{\eta}(p_f - p_g)^2$ by the strict convexity of the two functions.

Convexity Through Regularization

If **cumulative loss** $F_t(\cdot)$ was $1/\eta$-strongly convex (for all t), stability could be bounded as:

$$\sum_{t=1}^{T} |p_t - p_{t+1}| \leq \eta \cdot T \,,$$

because $F_t(p) - F_{t-1}(p) = f(p; \boldsymbol{\ell}_t)$ is 1-Lipschitz (due to $\boldsymbol{\ell}_t \in [0,1]^2$).

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But our cumulative loss $F_t(\cdot)$ **is not** strongly convex!

Make it strongly convex through **regularization** !

$\tilde{F}_t(p) = F_t(p) + R(p)/\eta$, where $R(\cdot)$ any 1-strongly convex function:

- $R(p) = p^2/2$
- $R(p) = p\ln(p) + (1-p)\ln(1-p)$
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Follow / Be the Regularized Leader

$F_t(p) = \sum_{\tau=1}^{t} f(p; \boldsymbol{\ell}_\tau)$ and $\tilde{F}_t(p) = \sum_{\tau=1}^{t} f(p; \boldsymbol{\ell}_\tau) + R(p)/\eta$

FTRL: $\tilde{p}_t = \arg\min_{p \in [0,1]} \tilde{F}_{t-1}(p)$

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Lower bound on $\text{Regret}_A(T)$ for any online (even randomized) optimization algorithm A?

Regret of FTRL Against BTRL

$$\text{Regret}_{FTRL}(T) \leq \text{Regret}_{BTRL}(T) + \sum_{t=1}^{T} |\tilde{p}_t - \tilde{p}_{t+1}|$$
$$\leq \text{Regret}_{BTRL}(T) + \eta \cdot T$$

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Proof : Second inequality from strong convexity, because $\tilde{p}_t, \tilde{p}_{t+1}$ are minimizers of $1/\eta$-strongly convex functions $\tilde{F}_{t-1}(p)$ and $\tilde{F}_t(p)$ with difference $f_t(p)$ which is 1-Lipschitz.

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$$\begin{aligned}\text{Regret}_{FTRL}(T) - \text{Regret}_{BTRL}(T) &= \sum_{t=1}^{T} (f(\tilde{p}_t; \boldsymbol{\ell}_t) - f(\tilde{p}_t^*; \boldsymbol{\ell}_t)) \\ &\leq \sum_{t=1}^{T} |\tilde{p}_t - \tilde{p}_t^*| \\ &= L \sum_{t=1}^{T} |\tilde{p}_t - \tilde{p}_{t+1}|\end{aligned}$$

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Proof : Let $f_t(p) = f(p; \boldsymbol{\ell}_t)$ for brevity.

- Let $f_0(p) = R(p)/\eta$ and $\tilde{p}_0^* = \arg\min_{p\in[0,1]} R(p)/\eta$.
- Using induction on t, we show that for all $t \geq 1$,

$$\sum_{\tau=0}^{t} f_\tau(\tilde{p}_\tau^*) \leq \tilde{F}_t(\tilde{p}_t^*) \quad \text{(including fake action } \tilde{p}_0^* \text{ at } \tau = 0\text{)}$$

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- Then, using the claim above,

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- Hence, by rearranging:

$$\sum_{t=1}^{T} f_t(\tilde{p}_t^*) - \min_{p\in[0,1]} \sum_{t=1}^{T} f_t(p) \leq \max_{p\in[0,1]} R(p)/\eta - \min_{p\in[0,1]} R(p)/\eta \leq \max_{p\in[0,1]} |R(p)|/\eta$$

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Multiplicative weight updates:

- Negative entropy $E^-(p) = p \ln(p) + (1-p)\ln(1-p)$ is 1-strongly convex wrt L_1 norm.
- Using $E^-(p)$ as regularizer, results in the following update rule for expected loss $f(p_t; \boldsymbol{\ell}_t) = p_t \ell_t^H + (1-p_t)\ell_t^L$:

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- If $\boldsymbol{\ell}_t \in [0,1]^2$, setting $\eta = \sqrt{\ln(2)/T}$, yields regret $2\sqrt{T \ln(2)}$