Constitutive equation of chemo-hyperelasticity

Stress expressions for the solid parts of the mixture.

$$\boldsymbol{\Sigma}^{gs} = \frac{\partial \underline{W}^{gs}}{\partial \boldsymbol{E}_G}, \quad \boldsymbol{\Sigma}^{c} = \frac{\partial \underline{W}^{c}}{\partial \boldsymbol{E}_G}$$

In terms of Cauchy stress

$$\boldsymbol{\sigma}^{gs} = \frac{1}{\det \boldsymbol{F}} \boldsymbol{F} \cdot \frac{\partial \underline{W}^{gs}}{\partial \boldsymbol{E}_G} \cdot \boldsymbol{F}^T$$

A quadratic form for $\underline{W}^{gs}$ in terms of $\boldsymbol{E}_G$

$$\underline{W}^{gs} = \frac{1}{2} \lambda (\mathrm{tr}\, \boldsymbol{E}_G)^2 + \mu \boldsymbol{E}_G : \boldsymbol{E}_G =$$

λ, μ: Lame constants

$\boldsymbol{E}_G$: may refer to large strains

$$= \frac{1}{8}(\lambda + 2\mu) I_1^2(\boldsymbol{C}) - \frac{1}{4}(3\lambda + 2\mu) I_1(\boldsymbol{C}) - \frac{1}{2}\mu I_2(\boldsymbol{C}) + \frac{3}{8}(3\lambda + 2\mu).$$

I_1, I_2: 1st and 2nd scalar invariants of the right Cauchy-Green deformation tensor $\boldsymbol{C}$.

$$I_1 = I_1(\boldsymbol{C}) = \mathrm{tr}\, \boldsymbol{C}$$

$$I_2 = I_2(\boldsymbol{C}) = \frac{1}{2}\left[I_1^2(\boldsymbol{C}) - \boldsymbol{C} : \boldsymbol{C}\right]$$

$$\boldsymbol{C} = 2\boldsymbol{E}_G + \boldsymbol{I}$$

$\boldsymbol{\Sigma}^{gs} = \dfrac{\partial \underline{W}^{gs}}{\partial \boldsymbol{E}_G}$ gives a linear stress-strain relation

If we use

$$\underline{W}^{gs} = \underline{W}_0^{gs} f$$

where $\underline{W}_0^{gs}$ is the quadratic form above and $f = f(\boldsymbol{C})$ a nonlinear scalar function of $\boldsymbol{C}$, we can make the $\underline{W}^{gs}$ strongly nonlinear, in order to get a non-linear stress-strain relation.

If $f = f(\underset{\sim}{C}) = \exp[\alpha(\det \underset{\sim}{C} - 1)]$
the stress-strain relation becomes nonlinear.
If $f = f(\underset{\sim}{C}) = 1$ we get linear stress-strain relation.

For the collagen fibrils, we may have (for a single fibril)

$$\underline{W}_c = \underline{W}_c(L^2)$$

$$L = \frac{1}{2}\langle \underset{\sim}{C} : \underset{\sim}{M}_c - 1 \rangle = \langle \underset{\sim}{E}_G : \underset{\sim}{M}_c \rangle$$

$\langle\ \rangle$: denote the positive part of the quantity inside

$\underset{\sim}{M}_c$: 2^{nd} order structure tensor related to the undeformed direction of the single collagen fibril

$$\underset{\sim}{M}_c = \underline{m}_c \otimes \underline{m}_c$$

$\underline{m}_c$: unit vector indicating the direction of a single collagen fiber in the undeformed configuration.

$\langle \underset{\sim}{E}_G : \underset{\sim}{M}_c \rangle$: a defomation (scalar) measure of the stretch of the fiber along its direction.

For example we can write

$$\underline{W}_c = \frac{\overline{K}_c}{2k_c}\left(\exp(k_c L^2) - 1\right)$$

where $\overline{K}_c, k_c$ appropriate stiffness constants

For a network of two fibers, we write

$$\underline{W}^c(\underset{\sim}{E}_G) = \sum_{i=1}^{2} \frac{\overline{K}_i^c}{2k_{ci}}\left[\exp(k_{ci} L_i^2) - 1\right]$$

where

$$L_i = \langle \underset{\sim}{E}_G : \underset{\sim}{M}_{ci} \rangle$$

Governing equations for the cornea

Nature	Field eq.	Unknowns
Momentum balance	$\operatorname{div} \underset{\sim}{\sigma} = \vec{0}$	Displacement $\vec{u}$
Mass balance	$\operatorname{div} \vec{v}_s + \operatorname{div} J_E$	Chemical potential of EF water g_{wE}

Mass balance of ions in the EF phase

$$\frac{1}{\det \underset{\sim}{F}} \frac{dN_{kE}}{dt} + \operatorname{div} \vec{J}_{kE} = 0$$

Electrochemical potentials of the kth ion: g^{ec}_{kE}

k: refers to the kth ion

N_{kE}: molar mass measure of the k^{th} ion in the Extrafibrillar phase

Vector of unknown quantities

$$\underset{\approx}{X} = \begin{bmatrix} \vec{u} \\ \hline g_{WE} \\ g^{ec}_{NaE} \\ g^{ec}_{ClE} \end{bmatrix}$$

3 + n_{sd} unkowns

Purely mechanical contribution with non-linear aspects

For a collagen fibril, the mapping from the undeformed to the deformed configuration, regarding its direction, is given by the relation

$$\vec{m} = \frac{\underset{\sim}{F} \cdot \vec{\underline{m}}}{|\underset{\sim}{F} \cdot \underline{m}|}$$

$\vec{m}$: directional unit vector of the collagen fibril in the deformed configuration

$\vec{\underline{m}}$: ---- in undeformed configuration

Accordingly, the 2^{nd} order structure tensors $\underset{\sim}{M}$ and $\underset{\sim}{\underline{M}}$ are related be the relation

$$\underset{\sim}{M} = \vec{m} \otimes \vec{m} = \frac{\underset{\sim}{F} \cdot \underline{M} \cdot \underset{\sim}{F}^T}{|\underset{\sim}{F} \cdot \underline{m}|^2}$$

For an one finaily of parallel fibers we have transverse isotropy (perpendicular to the single direction of the fibers)

The stain energy density function for a transversely isotropic material, is written in terms of the following strain invariants

$I_1 = \mathrm{tr}\, \underset{\sim}{C}$ $\underset{\sim}{C}$: Right Cauchy - Green deformation tensor

$I_2 = \frac{1}{2}\left[\mathrm{tr}^2 \underset{\sim}{C} - \mathrm{tr}\, \underset{\sim}{C}^2\right]$

$I_3 = \det \underset{\sim}{C}$

~~wts~~ Writing a $\underline{W}$ in terms of I_1, I_2, I_3 i.e $\underline{W} = \underline{W}(I_1, I_2, I_3)$ we refer to an isotropic material. For a transversely isotropic material, we use the following extra "invariants" (pseudo-invariants)

$I_4 = \underset{\sim}{C} : \underset{\sim}{M}$

$I_5 = \underset{\sim}{C}^2 : \underset{\sim}{M}$

A $\underline{W} = \underline{W}(I_1, I_2, I_3, I_4, I_5)$ is associated with a transversely anisotropic material, with pre- ~~ferred~~ fered anisotropy direction the $\underline{\vec{m}}$ (one indicated be $\underline{\vec{m}}$ where

$\underline{M} = \underline{\vec{m}} \otimes \underline{\vec{m}}$). The presence of I_3 in the form of $\underline{W}$ indicates a compressible material, like $\underline{W} = \underline{W}(I_1, I_2, I_3)$.

A form like $\underline{W} = \underline{W}(I_1, I_2)$ indicates an isotropic incompressible material.

Similarly a form like $\underline{W} = \underline{W}(I_1, I_2, I_4, I_5)$ indicates a transversely isotropic material, which is incompressible.

$I_3 \rightarrow$ relates to the volume change

$I_4 \rightarrow$ is related to the extension of the fibers along their direction

$I_5 \rightarrow$ no direct physical meaning, but is related to shear along directions perpendicular to the fiber direction.

For an incompressible and inextensible transversely isotropic material

$$\underline{W} = \underline{W}(I_1, I_2) + \frac{1}{2} p_F (I_3 - 1) + \frac{1}{2} T (I_4 - 1)$$

The stresses for this material are given by the formula:

$$\underline{\underline{\tau}} = 2\frac{\partial W}{\partial \underset{\sim}{C}} \Rightarrow \underline{\underline{\tau}} = 2\frac{\partial W}{\partial \underset{\sim}{C}} \underbrace{- p_F \det \underset{\sim}{C}\, C^{-1}}_{\text{Incopressibility term}} + \underbrace{T \underset{\sim}{M}}_{\text{Inextensibility term}}$$

The extra parameters p_F, and T are found ~~fr~~ from equilibrium equation of from boundary condition.