Growth law

$$-\underset{\sim}{f}_{\kappa\kappa} = \underset{\sim}{C}^{e}_{\kappa} : \frac{\partial E}{\partial \underset{\sim}{E}^{e}_{G\kappa}} + m^{\kappa} e_{\kappa} \underset{\sim}{I} + \underset{\sim}{B}^{(g)}_{\kappa\kappa}$$

elastic

free energy per unit mass?

tensor related to remodeling

kappa: refers to the growth stage (intermediate), before the elastic deformation.

For small strains we have

$$\underset{\sim}{\epsilon} = \underset{\sim}{\epsilon}^{e}_{\kappa} + e^{g}_{\kappa} \qquad g\text{: growth}$$

Additive partition of strains. A result of the multiplicative partition of the deformation gradient tensor $\underset{\sim}{F} = \underset{\sim}{F}^{e} \cdot \underset{\sim}{F}^{g}$

A type of $\underset{\sim}{E}$ (free energy) for growth

$$E(\underset{\sim}{E}_G, m) = m\, e(\underset{\sim}{E}_G, m) = \left(\frac{m}{m_h}\right)^{n_1} f(\underset{\sim}{E}_G)$$

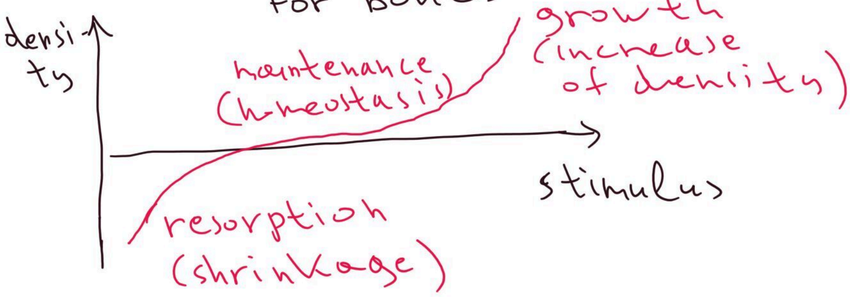
e: free energy per unit volume

depends to E, E_h

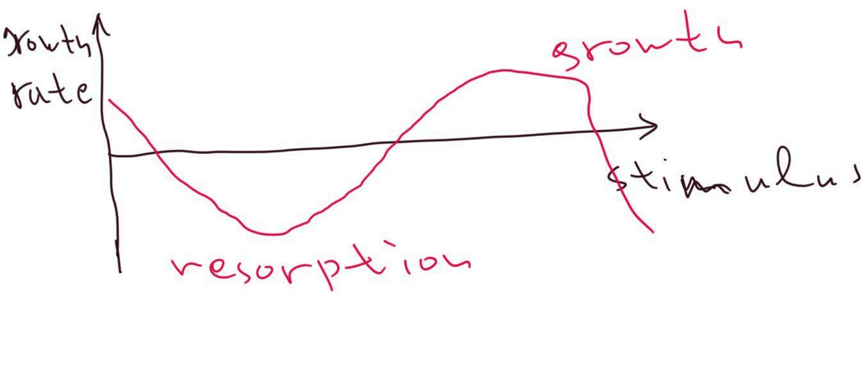
m: current mass

m_h: homeostatic mass

For bones



For soft tissues



Features of growth

1) May involve growth strain ~~d~~ or not.
2) Material elastic of viscoelastic
3) Single phase of mixture
4) Exchange of mass only on energy as well with the surrounding
5) Growth only or restructuring as well. (remodelling)

Example of constitutive law for ~~grow~~ an elasticall growing material

$$\underset{\sim}{\sigma} = m \underset{\approx}{E}(m) : \underset{\sim}{E}_G$$

$\underset{\approx}{E}$: 4th order stiffness tensor

$\underset{\sim}{E}_G$: 2nd order tensor of strain

accompanied by the mass exchang equation

$$\frac{dm}{dt} = a(m) + \underset{\sim}{A}(m) : \underset{\sim}{E}_G + \underset{\sim}{E}_G : \underset{\approx}{C}(m) : \underset{\sim}{E}_G$$

$a(m)$: scalar function of mass

$\underset{\sim}{A}(m)$: tensor (2nd order) function of mass

$\underset{\approx}{C}(m)$: 4th order tensor of mass

Growth models with growth strain

A viscoelastic model of a Maxwel isotropic, incompressible fluid

$$\underset{\sim}{\sigma} = -p \underset{\sim}{I} + \underset{\sim}{s} \quad ,$$

$\underset{\sim}{s}$: deviator of stress

Constitutive equation

$$\underset{\sim}{s} + t_r \frac{d\underset{\sim}{s}}{dt} = 2\mu t_r \left(\frac{d\underset{\sim}{\epsilon}}{dt} - \frac{d\underset{\sim}{\epsilon}^g}{dt} \right)$$

Deviatoric the rhs,

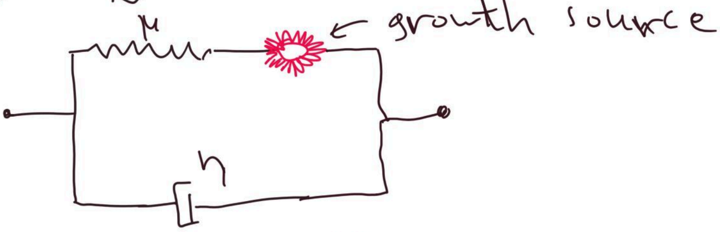
$$\underset{\sim}{\epsilon} - \underset{\sim}{\epsilon}^g = \underset{\sim}{\epsilon}^e$$

Viscoelastic meders involving growth

spring-dashpot-growth element

v: viscous

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^{e} + \underset{\sim}{\varepsilon}^{g} + \underset{\sim}{\varepsilon}^{v}$$



$$\varepsilon = \varepsilon^{e} + \varepsilon^{g} = \varepsilon^{v}$$

Example of constitutive equation for

- $\underset{\sim}{\varepsilon}^{g}$

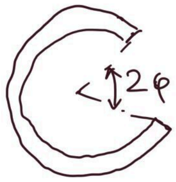
$$\frac{d\underset{\sim}{\varepsilon}^{g}}{dt} = \frac{\underset{\sim}{\alpha}_g}{t_g} : \left(\underset{\sim}{\sigma} + \underset{\sim}{\beta}\right)$$

t_g, $\underset{\sim}{\alpha}_g$ ← tensor growth parameter

characteristic time related to growth

Growth, accomodation, residual stress

Application to a circular, cylindrical artery wall



The angular displacement of the tips of the cut, indicate the presence of residual stresses in the intact artery

Stresses exist prior to application of loads, that keep the two tips of the cut glued together.

Reference configuration		Grown or intermediate configuration		Current configuration
	$\xrightarrow{\underset{\sim}{F}^g}$ grow (no stresses applied)		$\xrightarrow{\underset{\sim}{F}^e = \underset{\sim}{F}^a}$ a: assemble (glue again the tips of the cut together)	

$\underset{\sim}{\sigma} = 0$,	$\underset{\sim}{\sigma} = 0$	$\underset{\sim}{\sigma} \neq 0$
$\underset{\sim}{F} = \underset{\sim}{F}^e = \underset{\sim}{F}^g = \underset{\sim}{I}$	$\underset{\sim}{F} = \underset{\sim}{F}^g$	$\underset{\sim}{F} = \underset{\sim}{F}^e \cdot \underset{\sim}{F}^g$
	$\underset{\sim}{F}^e = \underset{\sim}{I}$	$\underset{\sim}{F}^e \neq \underset{\sim}{I}$
$\underline{B}(\underline{r}, \underline{\theta}, \underline{z})$	$B_e(r_e, \theta_e, z_e)$	$B(r, \theta, z)$

The total deformation gradient tensor

$$\underset{\sim}{F} = \mathrm{diag}\left[\lambda_r = \frac{\partial r}{\partial \underline{r}},\ \lambda_\theta = \frac{r}{\underline{r}},\ \lambda_z\right]$$

$$r = r(\underline{r}, t)$$

The intermediate components of generalized displacements, are

$$r_e = \underline{r},\quad \theta_e = \underline{\theta}\,\frac{\varphi_0}{\pi},\quad z_e = \lambda_{gz}\,\underline{Z}$$

Then

$$\underset{\sim}{F}^g = \mathrm{diag}\left[\lambda_{gr} = 1,\ \lambda_{g\theta} = \frac{\varphi_0}{\pi},\ \lambda_{gz}\right]$$

From the 2^{nd} to the 3^{nd} configuration

$$r = r(r_e), \quad \theta = \theta_e \frac{\pi}{\varphi_0}, \quad z = \lambda_{ez} z_e$$

$$\underset{\sim}{F}^e = \operatorname{diag}\left[\lambda_{er} = \frac{dr}{dr_e}, \lambda_{e\theta} = \frac{r}{r_e}\frac{\pi}{\varphi_0}, \lambda_{ez}\right]$$

$$\underset{\sim}{E}^e_G = \frac{1}{2}\left[(\underset{\sim}{F}^e)^T \cdot \underset{\sim}{F}^e - \underset{\sim}{I}\right] = \frac{1}{2}\operatorname{diag}\left[\lambda_{er}^2 - 1, \lambda_{e\theta}^2 - 1, \lambda_{ez}^2 - 1\right]$$

For incompressible material, $\det F^e = 1$, so the mapping ~~between the~~ from the 2^{nd} to the 3^{rd} configuration, in terms of radii, is given by

$$r^2 - r_1^2 = \left(r_e^2 - r_{e1}^2\right)\frac{\varphi_0}{\pi}\frac{1}{\lambda_{ez}} = \left(\underline{r}^2 - \underline{r}_1^2\right)\frac{\varphi_0}{\pi}\frac{1}{\lambda_{ez}}$$

r_1 = inner final radius

r: radius of any point in the wall in the final configuration

We have a single equilibrium equation (the others are satisfied identically)

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \quad (1)$$

The stress-strain relations are (for incompressible material)

k: for intermediate configuration

$$\underset{\sim}{\sigma} + p\underset{\sim}{I} = \operatorname{diag}\left[\frac{\partial E_k}{\partial \underset{\sim}{E}^e_{Gr}}\lambda_{er}^2, \frac{\partial E_k}{\partial \underset{\sim}{E}^e_{G\theta}}\lambda_{e\theta}^2, \frac{\partial E_k}{\partial \underset{\sim}{E}^e_{Gz}}\lambda_{ez}^2\right] \quad (2)$$

Then the equilibrium equation (1), from costitutive equations (2), yieds after integration

$$\sigma_r(r) - \underset{\overset{\parallel}{0}}{\sigma_r(r_1)} = \frac{\pi}{\varphi_0}\frac{1}{\lambda_{ez}}\int_{r_{e1}}^{r_e}(\sigma_r - \sigma_\theta)\frac{1}{\lambda_{e\theta}^2}\frac{dr_e}{r_e}$$

For $r_e = r_{e2}$, where $\sigma_r(r_2 = 0)$, the above relation yields the relation

$$Res(r_1) \equiv \int_{r_{e1}}^{r_{e2}} (\sigma_r - \sigma_\theta) \frac{1}{\lambda_{e\theta}^2} \frac{dr_e}{r_e} = 0$$

from which r_1 is found.

The indeterminate pressure, can be found from equilibrium equation as

$$p(r_e) = \frac{\partial E_K}{\partial E^e_{\sim Gr}} \lambda^2_{er} + \frac{n}{c_0} \frac{1}{\lambda_{e2}} \int_{r_{e1}}^{r_e} (\sigma_r - \sigma_\theta) \frac{1}{\lambda^2_{e\theta}} \frac{dr_e}{r_e}$$