Fair Division: Criteria and Algorithms for Allocating Indivisible Resources

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Outline

- **Part 1:**

  ➢ Basic models, and definitions of fairness criteria

- **Part 2:**

  ➢ Algorithms for relaxed versions of envy-freeness

- **Part 3:**

  ➢ Relations between (approximate) relaxations of envy-freeness

Cake-cutting problems

Input:

- ➢ A set of resources

- ➢ A set of agents, with possibly different preferences

likes everything equally

is allergic to strawberry

loves chocolate

Cake-cutting problems

Input:

➢ A set of resources

➢ A set of agents, with possibly different preferences

Goal: Divide the resources among the agents in a fair manner

Empirically: since ancient times

Mathematical formulations: Initiated by [Steinhaus, Banach, Knaster ʾ48]

Application domains

- Computer Science:
  - Resource allocation in computer systems
  - Job scheduling in operating systems
  - Task assignment in robot teams
- Operations research:
  - Load balancing problems, e.g. flight assignments to aircrafts
- Microeconomics:
  - Mechanism design for allocations of goods
- Political sciences:
  - Design of voting rules (for committee elections, seat allocations)

Available implementations: http://www.spliddit.org [Jonathan Goldman, Ariel Procaccia, CMU]
  - Algorithms for various problems (rent division, division of goods, etc)

Modeling fair division problems: the discrete setting

- This talk: focus on indivisible items

- Resources = a set of indivisible goods M = {1, 2, ..., m}

- Set of agents: N = {1, 2, ..., n}

- An allocation of M is a partition S = (S_1, S_2, ..., S_n), $S_i \subseteq$ M

  ➤ $\bigcup_i S_i$ = M and $S_i \cap S_j = \varnothing$

- Preferences: modeled by a valuation function for each agent

Valuation functions

All valuations we consider satisfy:

- $v_i(\varnothing) = 0$ (normalization)

- $v_i(S) \leq v_i(T)$, for any $S \subseteq T$ (monotonicity)

Special cases of interest:

➤ Additive: $v_i(S \cup T) = v_i(S) + v_i(T)$, for any disjoint sets S, T

  ➤ Assumed in the majority of the literature

  ➤ Suffices to specify v_{ij} for any good j: $v_i(S) = \Sigma_{j \in S} v_{ij}$, for any $S \subseteq M$

➤ Additive with identical rankings on the value of the goods

➤ Identical agents: Same valuation function for everyone

➤ Subadditive: $v_i(S \cup T) \leq v_i(S) + v_i(T)$, for any $S, T \subseteq M$

➤ Submodular: $v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T)$, for any $S \subseteq T$, and $j \notin T$

The discrete setting

Example with additive valuations

	TV	Skateboard	Computer	Book	Scooter
Charlie	35	5	25	0	35
Franklin	30	40	35	5	40
Marcie	30	20	40	30	0

Part 1: solution concepts for indivisible items

Solution Concepts

1. Proportionality

An allocation $(S_1, S_2,..., S_n)$ is proportional, if for every agent i,

$$v_i(S_i) \geq 1/n \cdot v_i(M)$$

Historically, the first concept studied in the literature

[Steinhaus, Banach, Knaster '48]

Solution Concepts

2. Envy-freeness

An allocation $(S_1, S_2, ..., S_n)$ is envy-free, if $v_i(S_i) \geq v_i(S_j)$ for any pair of players i and j

- Suggested as a math puzzle in [Gamow, Stern '58]
- More formally discussed in [Foley '67, Varian '74]

A stronger concept than proportionality (as long as valuations are subadditive)

Some issues

➢ These definitions are "too strong" for indivisible goods

➢ No guarantee of existence

➢ No guarantees of reasonable approximations either

➢ More appropriate for the continuous setting (existence is always guaranteed there)

➢ Need to explore relaxed versions of fairness

Solution Concepts

3. Envy-freeness up to 1 good (EF1)

An allocation $(S_1, S_2,..., S_n)$ satisfies EF1, if for any pair of agents i, j, there exists a good $g \in S_j$ such that $v_i(S_i) \geq v_i(S_j \setminus \{g\})$

- i.e., for any player who may envy agent j, there exists an item to remove from S_j and eliminate envy
- Defined by [Budish '11]

Solution Concepts

4. Envy-freeness up to any good (EFX)

An allocation $(S_1, S_2,..., S_n)$ satisfies EFX, if for any pair of agents i, j, and any good $g \in S_j$, we have $v_i(S_i) \geq v_i(S_j \setminus \{g\})$

- Removing any item from each player's bundle eliminates envy from other players
- Defined by [Caragiannis et al. '16]

Fact: Envy-freeness $\Rightarrow$ EFX $\Rightarrow$ EF1

Solution Concepts

5. Maximin Share Allocations (MMS)

A thought experiment:

- Suppose we run the cut-and-choose protocol for n agents.

- Say agent i is given the chance to suggest a partition of the goods into n bundles

- The rest of the agents then choose a bundle and i chooses last

- Worst case for i: he is left with his least desirable bundle

Solution Concepts

- Given n agents and S ⊆ M, the n-maximin share of *i* w.r.t. M is

$$\mu_i(n, S) = \max_{T \in \Pi(S)} \min_{T_j \in T} v_i(T_j)$$

- max is over all possible partitions of S into n bundles

- min is over all bundles of a partition $T = (T_1, T_2, ..., T_n)$

- For S = M, $\mu_i := \mu_i(n, M)$ = the maximin share of *i*

Solution Concepts

An allocation $(S_1, S_2,..., S_n)$ is a maximin share (MMS) allocation if for every agent *i*, $v_i(S_i) \geq \mu_i$

- Introduced by [Budish '11] (motivated by fairness properties of approximate *competitive equilibrium of equal incomes*)
- Fact: Proportionality $\Rightarrow$ MMS

Maximin shares

	📺	🛹	💻	📕	🛵	
Charlie	35	5	25	0	35	$\mu_1 = 30$
Franklin	30	40	35	5	40	$\mu_2 = 40$
Marcie	30	20	40	30	0	$\mu_3 = 30$

MMS vs EF1 (and vs EFX)

How do MMS allocations compare to EF1 and EFX?

➢ They are generally incomparable

➢ There exist EFX allocations that are not MMS allocations

➢ There exist MMS allocations that do not satisfy EF1 (hence not EFX either)

MMS vs EF1 (and vs EFX)

	TV	Skateboard	Computer	Book	Scooter	
Charlie	35	5	25	0	35	$\mu_1 = 30$
Franklin	30	40	35	5	40	$\mu_2 = 40$
Marcie	30	20	40	30	0	$\mu_3 = 30$

A MMS allocation that does not satisfy EF1
- Charlie envies Franklin even after removing any item from Franklin's bundle

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Solution Concepts

6. Pairwise Maximin Share Allocations (PMMS)

- Motivated by the definition of MMS allocations but moving towards envy-freeness

- Request an MMS-type guarantee but for pairs of agents

- Consider an allocation $S = (S_1, S_2,..., S_n)$

- We look at every pair of agents, say *i, j*

- Collectively they receive $S_i \cup S_j$ under S

- We demand that both i and j receive at least their MMS share for the items in $S_i \cup S_j$

Solution Concepts

An allocation $(S_1, S_2, ..., S_n)$ is a pairwise maximin share (PMMS) allocation if for every pair of agents *i, j,* $v_i(S_i) \geq \mu_i(2, S_i \cup S_j)$

- Introduced by [Caragiannis et al. '16]
- Fact 1: PMMS $\Rightarrow$ EFX
- Fact 2: PMMS and MMS are incomparable
  - There exist PMMS allocations that are not MMS allocations and vice versa

Solution Concepts

7. Groupwise Maximin Share Allocations (GMMS)

- Even further: A generalization of PMMS allocations [Barman et al. '18]

- Apply the idea of PMMS, not only for pairs but for any subset of agents

- An allocation (S_1, S_2,..., S_n) is a groupwise maximin share (GMMS) allocation if for every i, and every R $\subseteq$ N with i $\in$ R
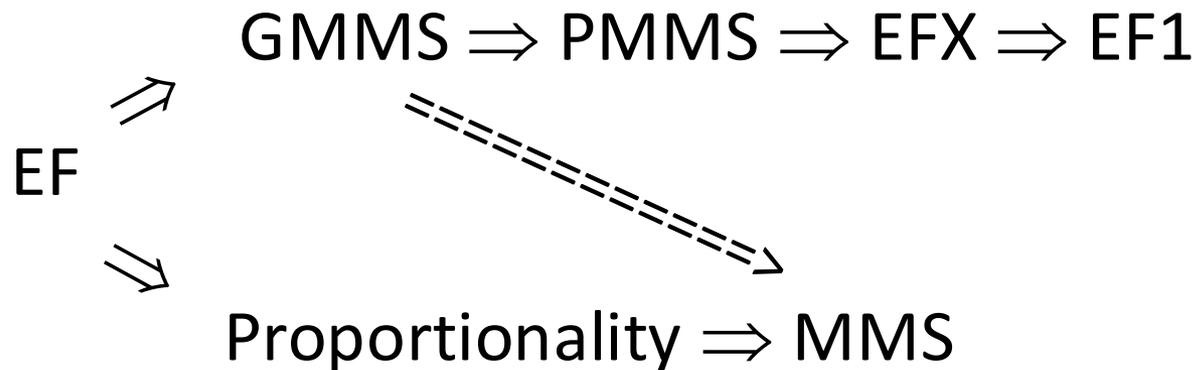
$$v_i(S_i) \geq \mu_i(|R|, \cup_{j \in R} S_j)$$

- Facts: GMMS $\Rightarrow$ PMMS, MMS, and EFX
  - The strongest of all relaxations

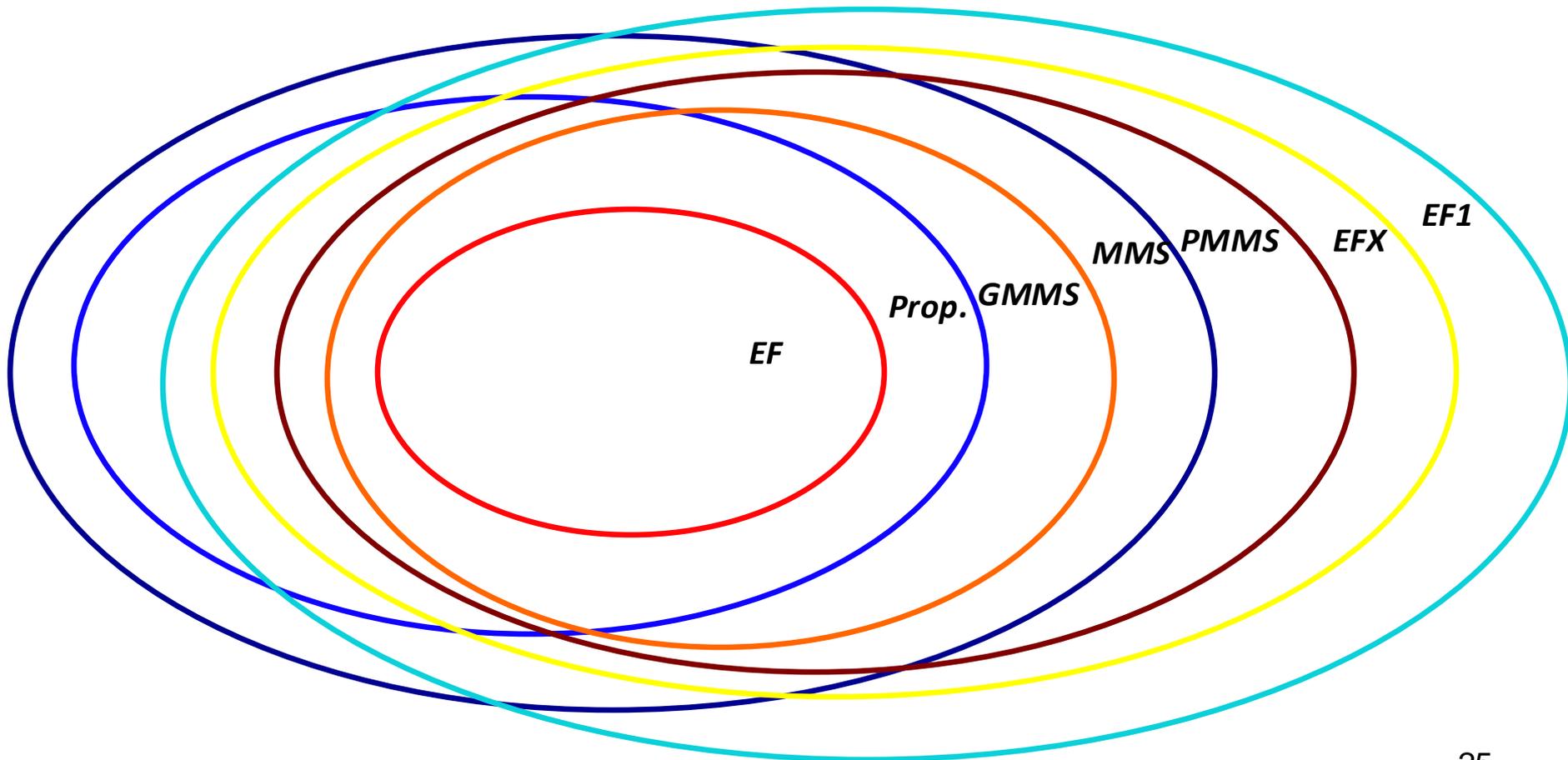# Relations between fairness criteria

For additive valuation functions
  ➢ Upper part holds for general monotone valuations

$$\text{GMMS} \Rightarrow \text{PMMS} \Rightarrow \text{EFX} \Rightarrow \text{EF1}$$

EF ↗

EF ↘

$$\text{Proportionality} \Rightarrow \text{MMS}$$

Relations between fairness criteria

Pictorially:

Approximate versions of solution concepts

- We are also interested in approximate versions of all the relaxations

- How can we define approximation here?
  - We focus on multiplicative approximations

- Example 1: An allocation $(S_1, S_2,..., S_n)$ is an α-EFX allocation, for α $\in$ [0, 1], if for any players *i* and *j*, and any good g $\in$ S_j, we have

$$v_i(S_i) \geq α\, v_i(S_j \setminus \{g\})$$

  - Similarly we can define α-EF1

- Example 2: An allocation $(S_1, S_2,..., S_n)$ is an α-MMS allocation if for every agent i, $v_i(S_i) \geq α\, \mu_i$

  - Similarly we can define α-PMMS and α-GMMS

Part 2: Existence and computation

EF1

Existence of EF1 allocations?

Theorem: For monotone valuation functions, EF1 allocations always exist and can be computed in polynomial time

- For additive valuations, a simple greedy round-robin algorithm suffices:
  - Select an arbitrary ordering of the agents
  - Each agent in his turn picks his favorite good among the currently available ones
- For non-additive valuations, more insightful to look at a graph-theoretic representation

EF1 for General Valuations

- Let S be an allocation (not necessarily of the whole set M)
- The envy-graph of S:
  - Nodes = agents
  - Directed edge (i, j) if i envies j under S

- How does this help?
  - Agents that nobody envies: in-degree = 0
  - Agents that at least one other person envies them: in-degree ≥ 1

EF1 for General Valuations

- An iterative algorithm till we reach a complete allocation
  - Suppose we have built a partial allocation that is EF1
  - If there exists a node with in-degree 0: give to this agent one of the currently unallocated goods
  - If this is not the case:
    - ➢ The graph has cycles
    - ➢ Start removing them one by one, by exchanging bundles, as dictated by each cycle
      - – each node in the cycle gets the bundle of the neighbor he is pointing to
    - ➢ Until we have a node with in-degree 0

EF1 for General Valuations

Algorithm 2 – The Cycle Elimination Algorithm

[Lipton, Markakis, Mossel, Saberi '04]

- Fix an arbitrary ordering of the goods, say, 1, 2, …, m
- At iteration i:
    - Find a node j with in-degree 0 (by possibly eliminating cycles from the envy-graph)
    - Give good i to agent j

Proof of correctness:
- Removing cycles terminates fast
    - Number of edges decreases after each cycle is gone
- At every step, we create envy only for the last item
- The allocation remains EF1 throughout the algorithm

Conclusion: EF1 is an "easy" concept to satisfy

EFX

Existence of EFX allocations?

- for n = 2
  - ➤ YES (for general valuations)
  - ➤ Variant of cut-and-choose
- for n=3
  - ➤ YES [Chaudhury, Garg, Melhorn '20]
- for n ≥ 4
  - ➤ Great open problem!
  - ➤ Guaranteed to exist only for agents with identical valuations or identical rankings over the goods
- Existence of approximately EFX allocations?
  - ➤ Recent positive results, best possible approximation not yet resolved

Algorithmic and complexity results for EFX allocations

[Plaut, Roughgarden '18]:

Separation between general and additive valuations

Theorem:

1. exponential lower bound even for 2 agents with identical submodular valuations

2. Polynomial time algorithm for 2 agents and arbitrary additive valuations

3. Polynomial time algorithm for any n, and additive valuations with identical rankings

   - All agents have the same ordering on the value of the goods

Algorithmic and complexity results for EFX allocations

Algorithm for additive valuations with identical rankings:

- Cycle elimination comes to rescue

- Run the cycle elimination algorithm

  - but first order the goods in decreasing order of value

➢ At every step of the algorithm we allocate the next item to an agent no-one envies

➢ Envy we create is only for the item at the current iteration

➢ But this has lower value than all the previous goods for all agents, since they have identical rankings

➢ Hence the allocation remains EFX throughout the algorithm

Approximate EFX allocations

[Plaut, Roughgarden '18]:

➢ Existence of ½-EFX for arbitrary additive valuations

    ➢ Extends also to subadditive valuations

➢ Again making use of the cycle elimination algorithm

➢ But with the option of de-allocating some items that were given to a player in previous steps

➢ Worst case exponential complexity

More recently:

[Chan et al. '19]:

➢ ½-EFX in polynomial time

Approximate EFX allocations

[Amanatidis, Markakis, Ntokos '20]:

➢ 0.618-EFX Poly-time algorithm for (0.618 = 1/φ)

➢ A careful combination of:

  ➢ Greedy round-robin

  ➢ The cycle elimination algorithm

➢ It is known that both algorithms produce EF1 allocations (but are bad w.r.t. EFX)

➢ Currently best known approximation for EFX

➢ It satisfies other fairness criteria as well (to be discussed later)

Approximate EFX allocations

A sketch of our algorithm:

1. Find a "good" ordering of the players

    - Simulate step 2: If an agent envies a predecessor by a large factor, he is moved to a high priority position in the ordering (these are the privileged agents)

2. Run one round of Greedy Round-Robin using the above ordering

3. Run a partial round of Greedy round-robin with the reverse ordering

    - Only for the non-privileged agents

4. Run the cycle elimination algorithm for the remaining goods

Summary on EFX allocations

➢ Severe hardness results for non-additive valuations

➢ Existence results for identical agents or agents with identical rankings

➢ 0.618-approximation for additive valuations

Main open questions:

➢ Do EFX allocations exist for arbitrary additive valuations?

   ➢ [Chaudhury, Garg, Melhorn '20]: YES for **n=3** (but pseudo-polynomial time algorithm)

➢ What is the best approximation we can have in polynomial time?

Some experimental evidence

[M.Sc. thesis of A. Ntokos '19]

➢ Series of experiments ran on synthetic data

➢ Data only for additive valuations

➢ Various distributions utilized for generating the value of every agent for each item

Some conclusions:

➢ EFX allocations existed in all generated instances

➢ As the number of items grows, the percentage of EFX allocations decreases

➢ As the number of items grows, Pr[an EFX allocation is also EF] increases

➢ We probably need more subtle constructions for finding counterexamples to existence (if they exist at all!)

MMS allocations

Existence?

- for n = 2

  ➢ YES (via a discrete version of cut-and-choose)

- for n ≥ 3

  ➢ NO [Procaccia, Wang '14]

  ➢ Known counterexamples build on sophisticated constructions

- How often do they exist for n ≥ 3 ?

  ➢ Actually extremely often

  ➢ Extensive simulations [Bouveret, Lemaitre '14] with randomly generated data did not reveal negative examples

Progress on α-MMS allocations

Established approximation guarantees for additive valuations:

- [Procaccia, Wang '14]

  ➢ 2/3-ratio, exponential dependence on n

- [Amanatidis, Markakis, Nikzad, Saberi '15]

  ➢ (2/3-ε)-ratio for any ε>0, poly-time for any n and m

- [Barman, Murty '17]

  ➢ 2/3-ratio, simpler algorithm, more intricate analysis

- [Garg, McGlaughlin, Taki '19]

  ➢ 2/3-ratio, simpler algorithm, simpler analysis

- [Ghodsi, Hajiaghayi, Seddighin, Seddighin, Yami '18]

  ➢ 3/4-ratio, poly-time but more complicated algorithm

2/3-approximation algorithms

The algorithm of [Barman, Murty '17]

Lemma 1: It suffices to establish the approximation ratio for additive valuations with identical rankings

For agents with identical rankings:

- Run the cycle elimination algorithm
  - but first order the goods in decreasing order of value
- We have seen that this is EFX

Lemma 2: This EFX allocation for additive valuations with identical rankings achieves also a 2/3-approximation for MMS

- Attention: not true in general for any EFX allocation

PMMS and GMMS allocations

Existence?

- for n = 2

  ➢ YES (both PMMS and GMMS are identical with MMS)

- for n ≥ 3

  ➢ Still an open problem

  ➢ Recall that PMMS implies EFX

  ➢ If we prove existence for either GMMS or PMMS, we have resolved the same question for EFX too

- How often do they exist for n ≥ 3 ?

  ➢ Very often (as with MMS and EFX)

  ➢ Simulations by [Barman et al. '18] did not reveal negative examples

Approximate PMMS and GMMS allocations

[Kurokawa '17]:

- A 0.781-PMMS algorithm
- Currently best known

[Barman et al. '18]:

- A ½-GMMS algorithm

- Yet another variant of cycle elimination:
  - Run the cycle elimination algorithm with no a priori ordering of the goods
  - in every step, determine an un-envied agent, as before
  - But now we give her the good she values the most out of the available ones
- This particular EF1 allocation is also a ½-GMMS allocation

Approximate GMMS allocations

Back to our work [Amanatidis, Markakis, Ntokos '20]:

Theorem 1: Our algorithm produces an allocation that is
- EF1
- 0.618-EFX
- 2/3-PMMS
- 0.553-GMMS (and same for MMS)

Theorem 2: We can tweak the initial algorithm to produce a 4/7-GMMS allocation in polynomial time
➢ 4/7-GMMS also achieved in [Chaudhury, Telikepalli, Mehlhorn, Sgouritsa, '20]
  - But in exponential time

Part 3: Relations between approximate versions of fairness

Approximately fair allocations

- We have already seen how the exact versions of fairness concepts are related to each other
- Does the picture change if we focus on the approximate versions of these concepts?
- Q: Does an allocation that is fair or approximately fair according to one notion implies any approximation guarantee regarding a different notion?
  - i.e. what does an α-MMS allocation imply about the other notions?

Approximately fair allocations

Why should we care?

- If we have an algorithm for approximating one notion, this may imply an approximation algorithm for another notion too

- Also technically interesting to understand which implications for exact versions carry over to the approximate versions

  - E.g., we know that a PMMS allocation is also EFX

  - Is an α-PMMS allocation also α-EFX?

Approximately fair allocations

- PMMS $\implies$ EFX

I found a PMMS allocation.

This is *awesome*,
because it is also EFX

Approximately fair allocations

α-PMMS ⇒ β-EFX ???

I found a 0.9-PMMS allocation.

This is *awesome*,
because I think it is also 0.9-EFX

Approximately fair allocations

α-PMMS ⇏ β-EFX

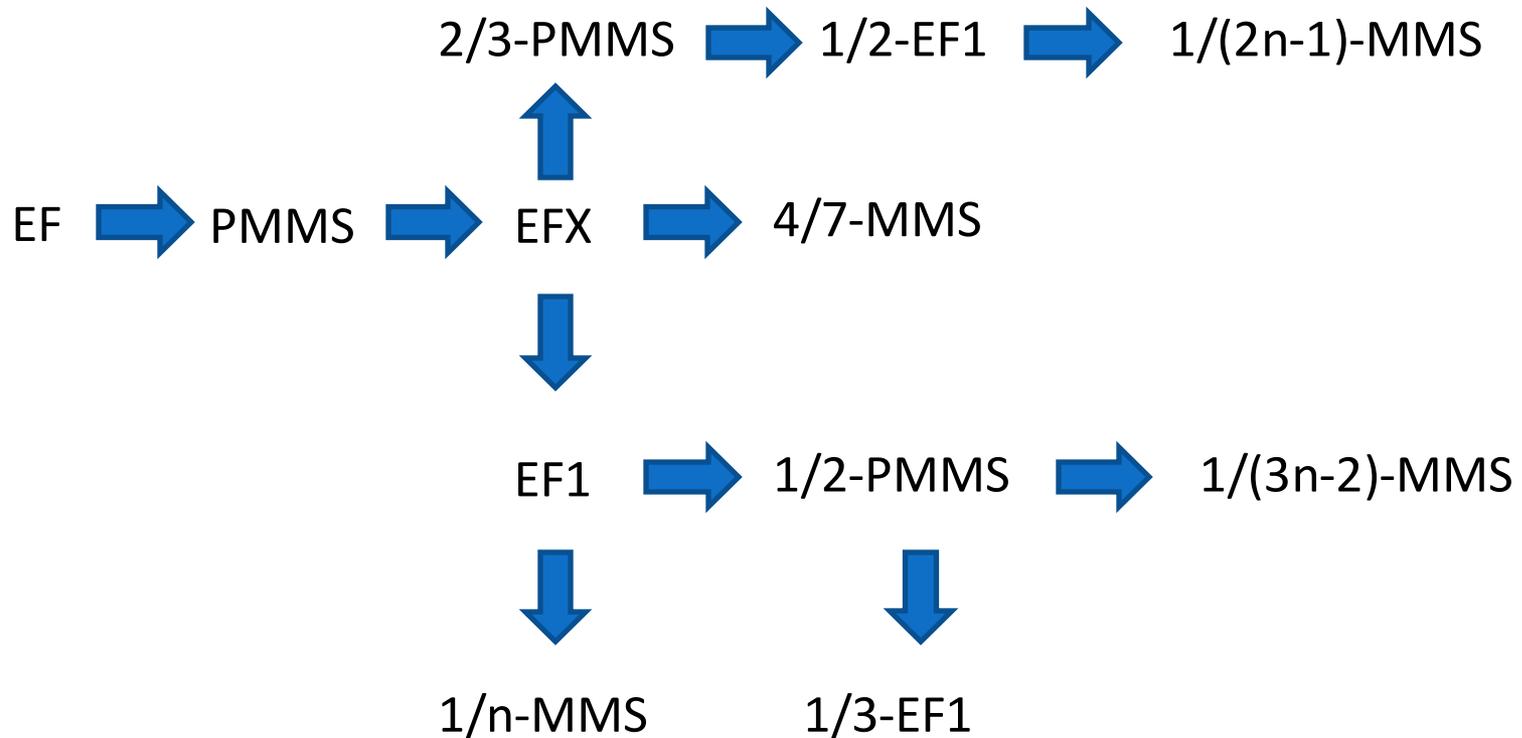I found a 0.9-PMMS allocation.

This is *awesome*,
because I think it is also 0.9-EFX

WRONG!!!

[Amanatidis, Birmpas, Markakis '18]:
A close to complete charting of the relationships between exact and approximate versions of EF1, EFX, MMS, PMMS for additive valuations

An Indicative Picture of the Landscape

2/3-PMMS → 1/2-EF1 → 1/(2n-1)-MMS

EF → PMMS → EFX → 4/7-MMS

EF1 → 1/2-PMMS → 1/(3n-2)-MMS

1/n-MMS   1/3-EF1

Highlights

Some representative results

- **Theorem 1:** Any EFX or PMMS allocation is also a 4/7(=0.57)-MMS allocation. On the other hand, an EFX allocation is not necessarily an α-MMS allocation for $\alpha > 0.59$

- **Theorem 2:** Any α-EF1 allocation is also an $\alpha/(1+\alpha)$-PMMS allocation for any $\alpha \in (0,1]$, and this is tight

- **Theorem 3:** An α-PMMS allocation is not necessarily a β-EFX allocation for any $\alpha, \beta \in (0, 1)$ [But PMMS $\Rightarrow$ EFX!]

- **Theorem 4:** An α-MMS allocation is not necessarily a β-PMMS, β-EF1, β-EFX allocation for any $\alpha, \beta \in (0, 1)$

Some implications

- By Theorem 2, EF1 $\Rightarrow$ ½-PMMS
  - Corollary: Greedy Round-Robin produces a ½-PMMS allocation
  - Although we know better approximations, this is a much simpler algorithm
- By Theorem 3, approximation algorithms for PMMS do not necessarily yield any approximation for EFX
  - MMS approximations also do not yield guarantees for EFX
- EFX approximations seem harder to get

Open problems and future directions

A rich area with several challenging ways to go
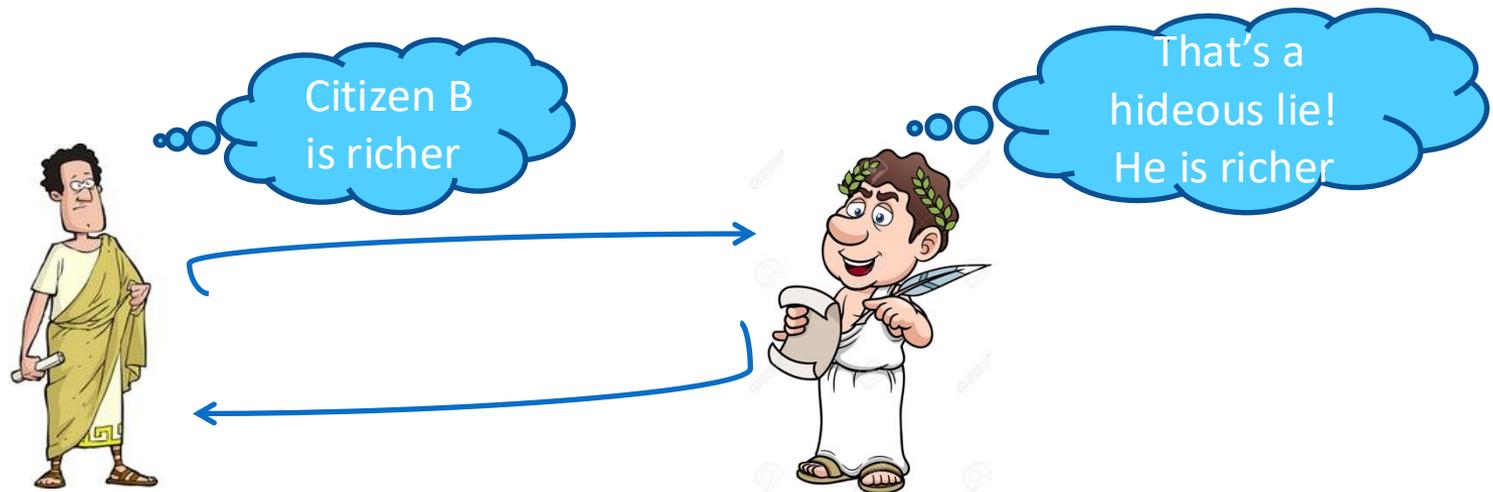
- Conceptual
  - Define or investigate further new notions and models
  - Fairness under mixed manna (desirable and undesirable goods)
  - Fairness in the presence of a social graph: evaluate fairness w.r.t. your neighbors [Chevaleyre, Endriss, Maudet '07, '17, Abebe, Kleinberg, Parkes '17, Bei, Qiao, Zhang '17, Aziz et al. '18]

- Algorithmic
  - Best approximation algorithm for the relaxed notions?
  - The continuous setting of fair division also has its own algorithmic challenges

- Game-theoretic
  - Mechanism design aspects?
  - So far progress mostly for n=2 [Amanatidis et al. '17]

Cycle elimination in ancient Greece: Αντιδοσις

A mechanism used for the sponsorship of theatrical performances

- The city of Athens would assign a wealthy citizen to cover the costs (for the chorus, costumes, instruments, etc)
- Suppose citizen A was chosen
- In case he wanted to appeal, he had to point to another wealthy citizen

Citizen B is richer

That's a hideous lie! He is richer

- If this happened, A and B had to exchange all their belongings!