Mathematics, Explanation, And Scientific Knowledge

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I

To prove the existence of a kind of entity is often to show how its existence helps explain a phenomenon. The molecular hypothesis is needed to explain the Brownian motion; the existence of light quanta, to explain the photoelectric effect. Let us ask first (a) whether physical phenomena have *mathematical* explanations, and (b) if so, whether any existential conclusions follow.

Consider this result, which has profound consequences for mechanics: the displacement of a rigid body about a fixed point can always be achieved by rotating the body a certain angle about a fixed *axis*. This result, first communicated by Euler, can be proved by a humdrum geometrical argument (see, for example, [14]: 2). To *explain* the result, however, we regard a rotation as a linear transformation of Euclidean space preserving distances and angles (scalar products), obtained continuously from the identity map (this eliminates reflections). We may further regard a linear transformation as a 3×3 matrix of real numbers. Now let A be any transformation and x a vector. If $Ax = \lambda x$, then λ is an *eigenvalue* of A; and x, an *eigenvector*. We then have the following theorem: if A is a rotation, then:

(a) the product of any eigenvalue with its complex conjugate is 1.

(b) the complex conjugate of an eigenvalue is an eigenvalue (the *complex conjugate* of the complex number $a + bi$, a and b real, is $a - bi$; note that we allow complex vectors even though the rotation matrix itself is real).

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(c) the product of all the eigenvalues is 1.

(See Goldstein [5], Chapter 4, for the proofs.) From (a) we derive that the square of a *real* eigenvalue is 1 (a real number is its own complex conjugate); thus each real eigenvalue must be ± 1. From (a) and (b), the product of all *complex* eigenvalues is $+1$. From (c), the product of all *real* eigenvalues must also be $+1$. Now λ is an eigenvalue of A if and only if the determinant of the matrix $A - \lambda I$ (obtained by subtracting λ from each element of the diagonal of A) is 0. Therefore the real eigenvalues are the real roots of a cubic equation, hence either one or three in number; their product, remember, being $+1$. Obviously one of the eigenvalues must be $+1$; therefore, there is a real vector x such that $Ax = x$; x, that is, is an axis of rotation.

Now (a), (b), and (c) do not depend on the 3-dimensionality of space. This is rather needed to demonstrate that the number of real eigenvalues must be odd. In two or four dimensions, the number of real eigenvalues is even; there is no necessity, then, for any eigenvalue to be $+1$. Thus, the explanation turns on the properties of odd versus even products of signed numbers. (Though I write "the" explanation, I deny that a phenomenon must have a unique explanation.)

This is, I stress, a *mathematical* explanation (of a physical fact), though obviously, physical assumptions enter: that physical space is a three-dimensional, Euclidean manifold. True, every scientific explanation requires mathematical as well as physical truths. Nevertheless, one senses a striking difference between this explanation and the previously mentioned explanation of the photoelectric effect.

The distinction between an explanation which uses mathematics and a characteristically mathematical explanation of a physical fact rests upon a proposition which I have argued elsewhere ([13])—that there are mathematical explanations of *mathematical* truth. Of two proofs of the same theorem, one may merely demonstrate the result, the second may explain it. Consider the following example, which I have not elsewhere cited: the formula

$$1/(1 + x^2) = 1 - x^2 + x^4 - x^6 + \dots$$

for x a real variable is valid whenever the series converges. We can prove that the series does not converge for $|x| = 1$; but we have no *explanation* of the nonconvergence, since the fraction $1/(1 + x^2)$ does not seem to behave badly as $|x|$ approaches 1 (indeed, early mathematicians were prone to say that the series

$$1 - 1 + 1 - 1 + \ldots$$

converges to ½). The "true" explanation appears only in the theory of *complex* variables, where Cauchy demonstrates that a power series in one complex variable must converge in a circular region about zero in the complex plane. Since the fraction $1/(1 + x^2)$—x now a complex variable—has a singularity when $x = i$, the series cannot converge when $x = 1$ (since the region would not take the shape of a circle). This example deserves close consideration, for it yields many rich insights (I am indebted to Ellis Kolchin for pointing it out to me).

I shall not reproduce my analysis of mathematical explanation here, but assume that mathematical explanation of *mathematical* truth exists. The difference between mathematical and physical explanations of *physical* phenomena is now amenable to analysis. In the former, as in the latter, physical and mathematical truths operate. But only in mathematical explanation is this the case: when we remove the physics, we remain with a mathematical explanation—of a mathematical truth! In our example, the "bridge" between physics and mathematics is the assumptions that space is three-dimensional Euclidean, and that the rotation of a rigid body around a point generates an orthogonal, real, proper transformation (to use the lingo). Deleting these assumptions, we obtain an explanatory proof of a theorem concerning transformations and eigenvectors. In standard scientific explanations, after deleting the physics nothing remains.

There are, then, mathematical explanations in physics. Do they make reasonable the existence of mathematical entities? If so, the explananda should not already be committed to such entities. But Goodman and Quine pointed out thirty years ago the apparent impossibility of describing the world without reference to numbers ([6]). I would put their point thus: to describe the experience of diversity and change requires mathematicatical entities. Imagine defining *rate of change* without the resources of analysis. We cannot say what the world would be like without numbers, because describing any thinkable experience (except for utter emptiness) presupposes their existence. (I owe this formulation to Sidney Morgenbesser.)

Quine's argument for the existence of numbers (which recurs again and again in Quine's later writings) is strangely Kantian. Even his argument for the existence of material bodies at the beginning of [10] can be interpreted as using the existence of the bodies to explain the properties of our sense data, though there is here, too, a whiff of Kantianism in Quine's remarks. Anyhow, many writers have remarked that in principle we could eliminate bodies and substitute local singularities in space-time. None of this holds for mathematical entities. Conclusion: no explanatory argument can establish the existence of mathematical entities—the Kantian argument makes evidence for the existence of numbers impossible.

This approach is far from those who argue from the Causal Theory of Knowledge, that empirical evidence for numbers is impossible because of the non-material qualities attributed to mathematical entities. Since Quine himself seems to favor a "naturalized" epistemology ([11]) receptive to a causal theory, tension could arise between the "Kantian" position and the view of the knowing subject as a passive recipient of information. Let us, then, see whether the Causal Theory of Knowledge is the epistemology required by modern science.

II

The form of the Causal Theory of Knowledge is not obvious. In *Mathematical Knowledge* ([13], Chapter 4) I distinguished two versions:

(a) One cannot know that a sentence S is true, unless S must be used in a causal explanation of one's knowing that S is true.

(b) One cannot know anything about F's unless this knowledge is caused by at least one event in which one F participates.

Relying on Davidson's distinction between the causal relation (involving events) and causal explanation (involving sentences) ([3]), I pointed out that version (a) is compatible with Platonism: the axioms of mathematics figure in any explanation of our knowledge of these axioms, for they form part of every scientific theory. Some have objected (e.g., Hart [7] : 124) that various *theorems* of mathematics will not so figure; but the Causal Theorist surely allows that any consequence of the known is known—otherwise the Causal Theory of Knowledge collapses in any event. (Consider predictions of future happenings, for example.) As for undecidable propositions, the Platonist could agree that they are *not* known—though for this answer to be credible, the axioms of arithmetic will need to be far more comprehensive than the ones codified in first-order Peano arithmetic. I emphasize that this is merely a *refutation* of an attack upon Platonism, not an explanation of mathematical knowledge.

The second version of the Causal Theory is better suited to attacking Platonism. An even more extreme version of (b) is

(c) It is impossible to know anything about an entity x unless x itself participates in the cause of that knowledge.

In his review of *Mathematical Knowledge*, W. H. Hart argues just this:

> Granted just conservation of energy, then, whatever your views on the mind-body problem, you must not deny that when you learn something about an object, there is a change in you. Granted conservation of energy, such a change can be accounted for only by some sort of transmission of energy from, ultimately, your environment to, at least proximately, your brain. And I do not see how what you learned about that object can be *about* that object (rather than some other) unless at least part of the energy that changed your state came from that object . . . .[what has to be explained is] how our beliefs could be about energetically inert objects. ([7]: 125).

Each sentence makes a separate point: the first two rely on physics; the third, upon reason. That Hart's case rests on natural science is no defect; it's the science which is defective:

1. Hart argues that every learner changes materially. But Hart implies that the change results from energy absorption. But the brain could change without disturbing its *net* energy.

2. The change in the learner need not be energy transmission *from* the environment. I am now sucking a piece of ice and learning while losing energy.

3. What is truly preposterous is the requirement that the energy originate in the known. Astrophysicists comb the universe for "black holes," bodies so dense that their emitted photons cannot escape; they are detectable only if they eclipse other bodies. An even more decisive objection (suggested by Richard Friedberg) arises from the motion of a charged particle in a constant magnetic field. The force on the particle is perpendicular to its velocity; the particle, therefore, gains no energy. Whatever energy is involved in detecting a magnet from the helical peregrinations of the particle does not come from the magnet.

4. It may be thought that these counter-examples arise from Hart's overspecialized view of the Causal Theory of Knowledge. The transfer of information, it will be argued, need not involve the transfer of energy. Other causal influences exist. But even this view is open to refutation.

Consider a stationary lithium 6 nucleus which suddenly undergoes beta decay, yielding an electron and a helium 6 nucleus traveling at right angles to one another. Since this by itself violates the conservation of momentum, there must be a third, undetected (because uncharged) particle (the neutrino) produced in the decay. (I am oversimplifying, of course, because there might as well be two such particles—but let us ignore this complication.) This particular argument is pictorially most striking (see [4]: 356).

Now we do not learn about the neutrino by transmission of energy from the neutrino to us—the neutrino is very difficult to detect by direct interaction. Indeed, as far as is known, beta decay is noncausal—no anterior event causes the breakup of the unstable lithium 6 nucleus. Nor does the neutrino participate in any event which causes the other particles' motion—through which we infer the existence of the neutrino. Beta decay "just happens" in accordance with the law of conservation of momentum, enabling us to infer a new particle. Laws of conservation are simply not causal laws. They provide constraints on what is allowed to happen.

It is true, however, that the neutrino participates in the beta decay which can be regarded as the "cause" of our discovery of the neutrino. Thus we can force this example to obey the Causal Theory of Knowledge, though the spirit of the theory is violated—for the known entity exerts no "causal influence" on the knower. But the scientific inference exemplified here is not limited to the case in which an explosion or a decay sets off a causal chain.

For a speculative example, consider Dirac's famous argument that if a single magnetic monopole exists anywhere on the universe, electric charge must be quantized—so that the existence of a particle with minimum (say) negative charge is in the cards. This amazing argument relies on both the conservation and the quantization of angular momentum. Now no one has ever discovered a magnetic monopole; and the electron is already discovered. But in principle, one could have used Dirac's argument and the discovery of a monopole to predict the electron, despite the total absence of any reference to a causal chain in the argument. In short, the Causal Theory of Knowledge depicts physical science very naively—to the extent that the theory is based on science at all.

5. As for the question (raised by Hart) as to how the reference of scientific terms is fixed—if not by some causal chain—the answer is, as Frege and Russell thought, by definite descriptions. Kripke ([9]) may well be right that ordinary proper names and many scientific terms fix their reference by causal chains, but discoveries in science are often introduced into the language as co-referential with descriptions.

(Actually, as Haim Gaifman points out to me, most of these examples merely illustrate the consequences for the Causal Theorist of the deductive closure condition mentioned before given this ability to introduce designators simply as coreferential with definite descriptions. Beginning with Pa, in virtue of energy received from a, we may still end up with Qb–b entirely disconnected from us—through the mediation of some theory T. This is essentially what happens in the Dirac example.)

The "Kantian" argument for the existence of mathematical entities is nothing like these examples of scientific inference; I have already explained that the existence of mathematical entities cannot be demonstrated by scientific inference. My only aim here is to show that the Causal Theory of Knowledge—often invoked to block mathematical knowledge—is not scientific. The Causal Theory of Knowledge regards all knowledge on the model of perception; but scientific inference cannot be so regarded.

We have discussed knowing the existence of number. Now I shall *assume* that numbers exist and ask whether mathematical truths concerning them are known *a priori*, or whether experiment could refute them.

III

Benacerraf is right that arithmetic is simply the science of progressions—if he means that neither mathematics nor science require more in the way of "objects" than a single infinite progression of objects of any character. For the second-order Peano postulates are categorical—only a progression models these axioms. Further, only one function on a progression satisfies the equations

$$x + 0 = x$$

$$x + Sy = S(x + y)$$

where S is the successor function of the progression. Benacerraf's view does not obviate ontology, since someone has to guarantee at least one progression—and it is irrelevant whether we call the guarantor "arithmetic" or something else. But since now we are assuming the existence of numbers (an assumption amounting to the existence of a progression) this is not our concern.

Could empirical evidence undermine arithmetic *without* undermining the existence of an infinite progression? One recalls the standard thought-experiments in which one counts 20 apples and 30 pears and reaches 51. By distinguishing between "pure" and "applied" mathematics, Körner ([8]) concludes that a standard answer—that we would always regard this as a mistake or hallucination—is wrong.

Körner's own solution—that we would revise arithmetic, at least "applied" arithmetic, is unnecessary (not false, since *something* in this example is falsifiable, and one can always call it applied arithmetic). All arithmetic identities, we have seen, follow from stipulations. Once a progression has been chosen, and the operations on this progression uniquely defined, any change in arithmetic is simply verbal (compare [2]). Consider the thought-experiment of counting apples and pears once again. When we say that 20 apples and 30 pears yield 50 fruits we mean:

(a) If the number of a set M of apples is x and the number of a disjoint set N of apples is y, then the number of the two sets is $x + y$.

(b) $20 + 30 = 50$.

Neither (a) nor (b) has "empirical content"; in order to apply such statements, we need the following "empirical" premises:

(c) Procedure A establishes the number of apples in M; and B, the number of pears in N.

(d) Procedure C establishes the number of fruits in $M \cup N$.

(These procedures will have to be described without presupposing arithmetic truths, or simply named without being described.) Now we have seen that (b) is simply true by stipulation, so long as we assume the existence of a progression. Statement (a) is proved by induction once we prove that if the number of M is n, and if object a is not in M, then the number of $M \cup \{a\}$ is Sn, the successor of n. Thus any deviation of experiment from arithmetic law must be laid to premises (c) or (d). Taking (c) for granted, we conclude that the failure of experiment to verify a sum by counting is the failure of a certain physical procedure to determine the number in the union of two disjoint sets. Some *other* procedure needs to be substituted, or else we can keep the procedure, associating it with some function other than addition—which is precisely what happens when one physical law is replaced by another. A society which calculated the square root of s by dropping a marble s feet, measuring the time and multiplying by 4 (according to the formula $s = 16t^2$), would have to change the method when s grew very large. Thus empirical results could cause us to declare arithmetic not false but inapplicable—it could happen that we need other operations than addition in the grocery store.

(Note that we have not investigated what kind of empirical evidence, if any, might help decide unidecidable propositions: those whose truth or falsity is unprovable from known axioms of arithmetic or set theory. We have studied truths already known to follow from the first-order Peano axioms—and atomic propostions at that.)

Since the "natural events" which addition predicts involve counting, there are two apparent disanalogies between the case of addition and the imaginary case of the last paragraph. The first is that counting is felt to be the historical *criterion* for finding the sum of two groups of objects. It is part of the meaning of '+' that counting should give the answer. But Kripke's analysis of reference ([9]) shows this objection to be baseless. Counting is used to fix the reference of '+', not to give it meaning. Once '+' is stipulated, however, it keeps its reference in all possible worlds—including those in which counting deviates, in which one needs another function than addition to predict the outcome of counting. (Note that in any event counting is not generally used to check numerical operations; in most cases the numbers are too large.)

A more subtle consideration also feeds the belief that counting is privileged, that addition is defined as the result of counting. Namely, it is impossible to *imagine* a case in which counting, across the board, should conflict with theorems of arithmetic. For if whenever we counted two disjoint groups of physical objects containing 7 and 5 objects respectively, the total count were 13, we would also conclude that $7 + 5 = 12$ is no theorem and that, rather, $7 + 5 = 13$ is. For to know what we are proving, we count pencil marks (compare [15], Part II). Deducing arithmetic truths from axioms involves the very counting skills that our deductions are to check! Counting and deducing, then, can never be *perceived* to conflict.

Even if this argument showed that arithmetic could be falsified, the same argument shows that the falsification could never be detected. So we could never know whether or not arithmetic has been falsified. Anyhow, counting is not the only way of deducing theorems from Peano's axioms. As I pointed out in *Mathematical Knowledge* ([12] :43-5), the decimal system of notation represents each number as a polynomial in 10. In doing additions in the decimal system, one need only know how to add up to 9 + 9—the rest is the arithmetic of polynomials which is free of "counting." Thus the point under consideration is valid only for additions up to 9 + 9. If we used the binary notation, we wouldn't have to count at all.

Finally, the argument might also be taken as showing, not that physical counting could correct arithmetic, but that the nature which would play tricks with counting would play tricks with our ability to catch the discrepancy.

In sum, arithmetic truths cannot be falsified, once we assume a single infinite progression. The existence of such a progression, however, cannot be experimentally demonstrated: Quine's argument for such a progression is uniquely Kantian, not (as in other cases) empirical. This conclusion is not disturbed by the existence of mathematical explanations for many physical events. Finally, the Causal Theory of Knowledge fails to threaten mathematical knowledge, for the simple reason that (in the forms discussed in this paper) it is false.

I myself have argued for continuity between the natural and mathematical sciences in *Mathematical Knowledge*. But such continuity begins and ends with methodological alikeness: both describe an objective world of entities, and (I argue) the methods used in exploring the two worlds are, despite common opinion to the contrary, remarkably similar. There may even be a power of observing mathematical truth akin to physical perception. But the foregoing considerations preclude any interaction between the two worlds.[1]

References

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NOTE

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