Functional Analysis I
(Problem sheet 7)

Exercise 1. *Show that the inner product*

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$$

is continuous.

Exercise 2. *Let H be an inner product space. Show that if for a sequence (x_n) we have that $\lim \|x_n\| = \|x\|$ and $\lim \langle x_n, x \rangle = \langle x, x \rangle$, then*

$$\lim x_n = x\,.$$

Exercise 3. *Show that*

$$Y = \big\{ x = (x_n) \in l^2 : \, x_{2n} = 0, \, n \in \mathbb{N} \big\}$$

is a closed subspace of l^2 and find $Y^\perp$.

Exercise 4. *Let M, N be closed subspaces of the Hilbert space H and suppose that $M \perp N$. Show that the set*

$$M + N = \{ x + y : \, x \in M \, , y \in N \}$$

is also a closed subspace of H.

Exercise 5. *Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal set in the Hilbert space H. Show that*

$$\|x - \sum_{k=1}^{n} \lambda_k e_k\| \geq \|x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k\| \,,$$

for all $\lambda_1, \lambda_2, ... \lambda_n \in \mathbb{R}$.

Exercise 6. *Find the first three functions that we get after applying the Gram-Schmidt method on the set $\{x^n : \, n = 0, 1, ...\}$ of $C[-1,1]$ with $\| \cdot \|_2$. In the sequel find numbers α, β and γ that miimize the quantity*

$$\int_{-1}^{1} |x^4 - \alpha - \beta x - \gamma x^2|^2 \, dx\,.$$

Exercise 7. *Let $\{e_n\}$ be an orthonormal basis in the Hilbert space H. Show that*

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \langle y, e_n \rangle \,,$$

for all $x, y \in H$.