Functional Analysis I
(Problem sheet 5)

Exercise 1. *Let X be a normed space. Show that for any $x \in X$ there exists $f \in X^*$, such that $f(x) = \|x\|^2$ and $\|f\| = \|x\|$.*

Exercise 2. *Let X be a normed space and Y be a proper closed linear subspace of X. If $x_0 \notin Y$ show that there exists $f \in X^*$, such that*

$$\|f\| = \frac{1}{d(x_0, Y)}\,, f(x_0) = 1 \text{ and } f(y) = 0, \text{ for all } y \in Y\,.$$

Exercise 3. *Let M be a subset of the nornmed space X. Show that $x_0 \in X$ belongs to the set $\overline{\langle M \rangle}$ if and only if $f(x_0) = 0$, for all $f \in X^*$, such that $f|_M = 0$.*

Exercise 4. *Let X be a normed space and Y be a closed linear subspace of X. We say that $z \in X$ is orthogonal to Y, and write $z \perp Y$, if $\operatorname{dist}(z, Y) = \|z\|$. Show that $z \perp Y$ if and only if there exists $0 \neq f \in X^*$ such that $f|_Y = 0$ and $|f(z)| = \|f\|\|z\|$. Moreover show that if $Y \neq X$ and $x_0 \notin Y$, then $y_0 \in Y$ is the nearest point of Y to x (i.e. $0 < \operatorname{dist}(x_0, Y) = \|x_0 - y_0\|$) if and only if $x_0 - y_0 \perp Y$.*

Exercise 5. *Let X be a normed space, Y be a linear subspace of X and*

$$F = \{f \in X^* : \|f\| \leq 1\,, f|_Y = 0\}\,.$$

Prove that:

(i) *For any $x \in X$*
$$\operatorname{dist}(x, Y) = \sup_{f \in F} |f(x)|\,.$$

(ii)
$$\overline{Y} = \bigcap_{f \in F} \ker f\,.$$

(Hint: use exercise 2.)