Functional Analysis I
(Solutions of problem sheet 4)

Exercise 1. *Let $T \in B(X,Y)$. Show that*

(i)
$$\begin{aligned} \|T\| &= \sup\{\|Tx\| : \|x\| \le 1\} \\ &= \sup\{\|Tx\| : \|x\| = 1\} . \end{aligned}$$

(ii)
$$\|T\| = \inf\{M > 0 : \|Tx\| \le M\|x\| , \text{ for all } x \in X\} .$$

Solution.

(i) We have that
$$\sup_{x\neq 0}\frac{\|Tx\|}{\|x\|} = \sup_{x\neq 0}\|T(\frac{x}{\|x\|})\| \le \sup_{\|x\|=1}\|Tx\| \le \sup_{\|x\|\le 1}\|Tx\| \le \sup_{\|x\|\le 1}\frac{\|Tx\|}{\|x\|} \le \sup_{x\neq 0}\frac{\|Tx\|}{\|x\|} .$$
Hence
$$\sup_{\|x\|=1}\|Tx\| = \sup_{\|x\|\le 1}\|Tx\| = \sup_{x\neq 0}\frac{\|Tx\|}{\|x\|} = \|T\| .$$

(ii) The set
$$\{M > 0 : \|Tx\| \le M\|x\| , \text{ for all } x \in X\}$$
consists of the upper bounds of the set
$$A = \left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} .$$
The result follows since by definition $\|T\|$ is the least upper bound of A.

□

Exercise 2 (Multiplication operator)**.** *Consider $X = C[a,b]$ equipped with $\|\cdot\|_\infty$ and $g \in X$. Define $T : X \to X$ by*
$$Tf(x) = g(x)f(x) , \text{ for all } x \in [a,b] .$$
Show that $T \in B(X)$.

Solution. If $f, h \in X$ and $\lambda \in \mathbb{R}$, then
$$T(f+h)(x) = (f+h)(x)g(x) = f(x)g(x) + h(x)g(x) = Tf(x) + Th(x)$$
and
$$T(\lambda f)(x) = (\lambda f)(x)g(x) = \lambda f(x)g(x) = \lambda Tf(x)$$
and therefore T is linear. Moreover we have that
$$\|Tf\|_\infty = \sup_{x\in[a,b]}|f(x)g(x)| \le \|g\|_\infty\|f\|_\infty$$
and hence T is bounded with $\|T\| \le \|g\|_\infty$. □

Exercise 3 (Right shift operator)**.** *Let $S : l^1 \to l^1$ defined by*

$$S(x_1, x_2, ...) = (0, x_1, x_2, ...) .$$

Show that S is bounded and find its norm.

Solution. If $x = (x_n) \in l^1$, then

$$\|Sx\|_1 = \sum_{n=1}^{\infty} |x_n| = \|x\|_1$$

and hence S is bounded with $\|S\| = 1$. □

Exercise 4. *Show that if X is a finite dimensional normed space, then every operator $T : X \to Y$, where Y is a normed space, is bounded.*

Solution. Assume that $\dim X = n$ and $\{e_1, e_2, ..., e_n\}$ is a basis of X. We know that all norms on X are equivalent and hence without loss of generality we will use $\|\cdot\|_2$ which for $x = \sum_{i=1}^{n} \lambda_i e_i$ is

$$\|x\|_2 = (\sum_{i=1}^{n} \lambda_i^2)^{\frac{1}{2}} .$$

If $\|\cdot\|$ is the norm of Y then

$$\|Tx\| = \|\sum_{i=1}^{n} \lambda_i e_i\| \leq \sum_{i=1}^{n} |\lambda_i| \|Te_i\| \leq (\sum_{i=1}^{n} \|Te_i\|^2)^{\frac{1}{2}} \|x\|_2$$

and hence T is bounded (we have used the Cauchy-Schwarz inequality). □

Exercise 5. *Let X be a Banach space.*

(i) *Suppose that $T \in B(X)$ and for all $y \in X$, the series*

$$\sum_{n=1}^{\infty} \|T^n y\|$$

converges. Show that for all $y \in X$, the equation

$$x = y + Tx$$

has a unique solution.

(ii) *We suppose that $T \in B(X)$ and $\|T\| < 1$. Show that the operator $I - T$ is "1-1" and onto.*

(iii) *Let $T : l^\infty \to l^\infty$ defined by*

$$T(x_1, x_2, x_3, ...) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, ...)$$

Show that for all $y \in l^\infty$, the equation

$$x = y + Tx$$

has a unique solution.

Solution.

(i) We will find the solution as the limit of the sequence (x_n) which is defined as follows:

$$x_0 \in X \text{ and } x_n = y + Tx_{n-1}, n \in \mathbb{N}. \tag{1}$$

For $n > m$ we have

$$\|x_n - x_m\| \leq \|T^m y + \dots + T^{n-1} y\| + \|T^n x_0\| + \|T^m x_0\|.$$

The first term on the right hand side converges to 0, as $n, m \to \infty$ since it is part of the tail of the convergent series

$$\sum_{n=1}^{\infty} \|T^n y\|.$$

Moreover by the convergence of $\sum_{n=1}^{\infty} \|T^n x_0\|$ the rest two terms also converge to 0. Hence the sequence (x_n) is Cauchy and since X is a Banach space, it converges to some $x \in X$. But then because of the continuity of T taking limits in (1) we have that

$$x = y + Tx.$$

For the uniqueness we note that because of the hypothesis there is no $x \neq 0$ with $Tx = x$, since then the series $\sum_{n=1}^{\infty} \|T^n x\|$ would be equal to $\sum_{n=1}^{\infty} \|x\|$ which obviously diverges. If x_1, x_2 are two solutions we have that

$$x_1 - Tx_1 = x_2 - Tx_2 \implies T(x_1 - x_2) = x_1 - x_2 \implies x_1 = x_2.$$

(ii) For all $x \in X$ we have that

$$\|T^n x\| \leq \|T\|^n \|x\|.$$

Hence the general term of the series $\sum_{n=1}^{\infty} \|T^n x\|$ is dominated by the general term of a geometric series with ratio $\|T\| < 1$ and hence by the comparison test the series $\sum_{n=1}^{\infty} \|T^n x\|$ converges. Hence the hypothesis of (i) is satisfied and so the equation $x = y + Tx$ has a unique solution for all $y \in X$. But this implies that $I - T$ is "1-1" and onto.

(iii) we have that

$$\|Tx\|_\infty = \sup_{n \geq 2} \frac{1}{n} |x_n| \leq \frac{1}{2} \|x\|_\infty.$$

Hence $\|T\| \leq \frac{1}{2} < 1$ and the result follows from (ii).

□