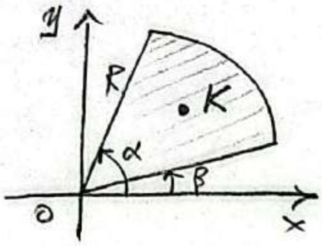
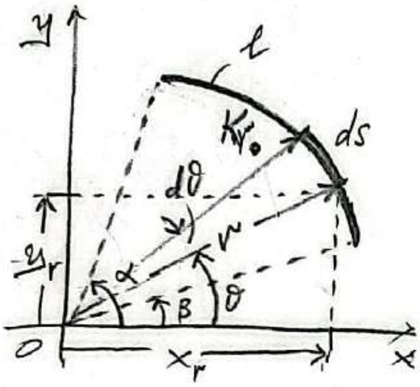
25η ΑΣΚΗΣΗ



Να υπολογισθούν οι συντεταγμένες x_K, y_K του κεντροειδούς K, του κυκλικού τομέα

1ος τρόπος Πρώτα υπολογισμός $K_r(x_{Kr}, y_{Kr})$ καμπύλης l.



$$ds = r\,d\theta,\quad x_r = r\cos\theta,\quad y_r = r\,\eta\mu\theta$$

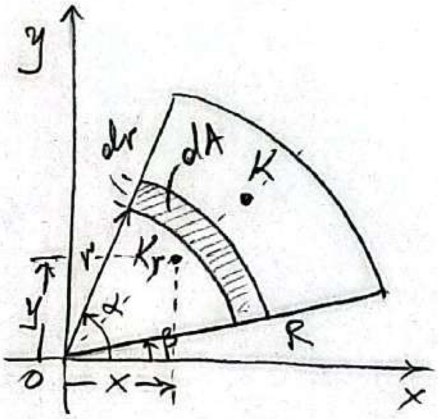
$$\boxed{l = \int_l ds = r\int_\beta^\alpha d\theta = r(\alpha-\beta)}$$

(Τα α, β με τα πρόσημά τους! ↶ + ↷ −)

$$\boxed{S_{y_r} = \int_l x_r\,ds = \int_\beta^\alpha r\cos\theta\cdot r\,d\theta = r^2\sin\theta\Big|_\beta^\alpha = r^2(\sin\alpha - \eta\mu\beta)}$$

$$\boxed{S_{x_r} = \int_l y_r\,ds = \int_\beta^\alpha r\,\eta\mu\theta\cdot r\,d\theta = r^2(-\cos\theta)\Big|_\beta^\alpha = r^2(\cos\beta-\cos\alpha)}$$

$$\boxed{x_{Kr} = \frac{\int_l x\,ds}{\int_l ds} = \frac{S_{y_r}}{l} = \frac{\sin\alpha-\eta\mu\beta}{\alpha-\beta}r},\quad \boxed{y_{Kr} = \frac{\int_l y\,ds}{\int_l ds} = \frac{S_{x_r}}{l} = \frac{\cos\beta-\cos\alpha}{\alpha-\beta}r}$$



Αντιστοιχία

$$dA = l\cdot dr = (\alpha-\beta)r\,dr$$

$$x = x_{Kr} = \frac{\sin\alpha-\eta\mu\beta}{\alpha-\beta}r$$

$$y = y_{Kr} = \frac{\cos\beta-\cos\alpha}{\alpha-\beta}r$$

$$\boxed{A = \int_A dA = (\alpha-\beta)\int_0^R r\,dr = \frac{\alpha-\beta}{2}R^2\ cm^2}$$

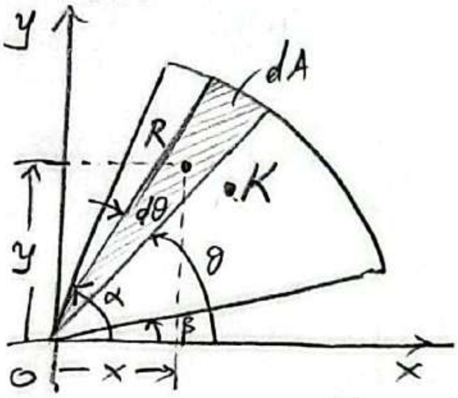
$$\boxed{S_y = \int_A x\,dA = \int_0^R \frac{\sin\alpha-\eta\mu\beta}{\alpha-\beta}r(\alpha-\beta)r\,dr = \frac{\sin\alpha-\eta\mu\beta}{3}R^3\ cm^3}$$

$$\boxed{S_x = \int_A y\,dA = \int_0^R \frac{\cos\beta-\cos\alpha}{\alpha-\beta}r(\alpha-\beta)r\,dr = \frac{\cos\beta-\cos\alpha}{3}R^3\ cm^3}$$

$$\boxed{x_K = \frac{\int_A x\,dA}{\int_A dA} = \frac{S_y}{A} = \frac{2(\sin\alpha-\sin\beta)}{3(\alpha-\beta)} R \text{ cm}}$$

$$\boxed{y_K = \frac{\int_A y\,dA}{\int_A dA} = \frac{S'_x}{A} = \frac{2(\cos\beta-\cos\alpha)}{3(\alpha-\beta)} R \text{ cm}}$$

2ος τρόπος



$$dA = \frac{1}{2} R \cdot R\,d\theta = \frac{R^2\,d\theta}{2}$$

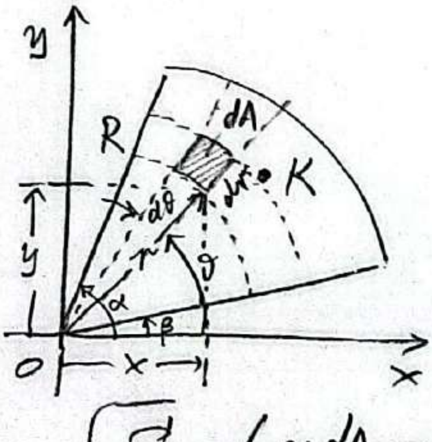
$$x \cong \frac{2}{3} R\cos\theta, \quad y \cong \frac{2}{3} R\sin\theta$$

$$\boxed{A = \int_A dA = \frac{R^2}{2}\int_\beta^\alpha d\theta = \frac{\alpha-\beta}{2} R^2 \text{ cm}^2}$$

$$\boxed{S_y = \int_A x\,dA = \int_\beta^\alpha \frac{2}{3}R\cos\theta\,\frac{R^2\,d\theta}{2} = \frac{R^3}{3}\sin\theta\Big|_\beta^\alpha = \frac{\sin\alpha-\sin\beta}{3}R^3 \text{ cm}^3}$$

$$\boxed{S_x = \int_A y\,dA = \int \frac{2}{3}R\sin\theta\,\frac{R^2\,d\theta}{2} = \frac{R^3}{3}(-\cos\theta)_\beta^\alpha = \frac{\cos\beta-\cos\alpha}{3}R^3 \text{ cm}^3} \ \ldots$$

3ος τρόπος



$$dA = r\,d\theta\,dr, \quad x = r\cos\theta, \quad y = r\sin\theta$$

$$\boxed{A = \int_A dA = \int_0^R r\,dr\int_\beta^\alpha d\theta = \frac{R^2}{2}(\alpha-\beta) \text{ cm}^2}$$

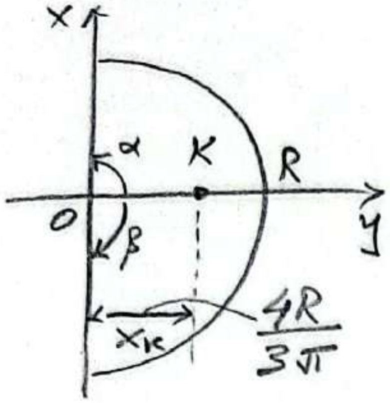
$$\boxed{S_y = \int_A x\,dA = \int_0^R r^2\,dr\int_\beta^\alpha \cos\theta\,d\theta = \frac{R^3}{3}(\sin\alpha-\sin\beta) \text{ cm}^3}$$

$$\boxed{S_x = \int_A y\,dA = \int_0^R r^2\,dr\int_\beta^\alpha \sin\theta\,d\theta = \frac{R^3}{3}(-\cos\theta)_\beta^\alpha = \frac{R^3}{3}(\cos\beta-\cos\alpha) \text{ cm}^3}$$

...

(1-22)

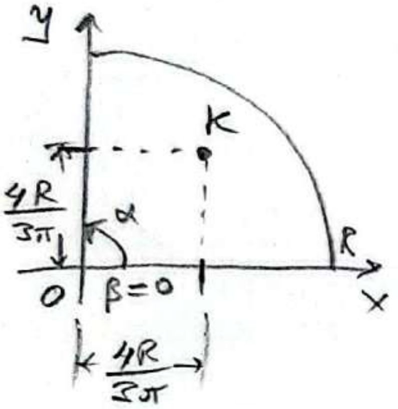
• Στην ειδική περίπτωση $\alpha = \frac{\pi}{2}$, $\beta = -\frac{\pi}{2}$:



$$\boxed{x_K = \frac{2\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right)}{3\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)} R = \frac{2(1+1)}{3\pi} R = \frac{4R}{3\pi}}$$

$$\boxed{y_K = \frac{2\left(\cos\left(-\frac{\pi}{2}\right) - \cos\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)} R = \frac{2 \times 0}{3\pi} = 0}$$

• Στην ειδική περίπτωση $\alpha = \frac{\pi}{2}$, $\beta = 0$:



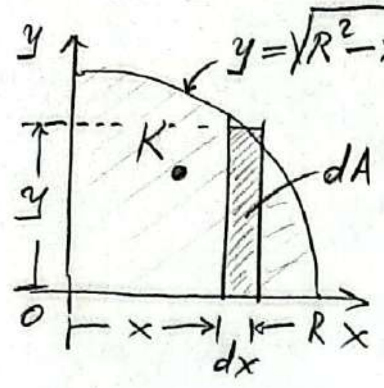
$$\boxed{x_K = \frac{2\left(\sin\frac{\pi}{2} - \sin 0\right)}{3\left(\frac{\pi}{2} - 0\right)} R = \frac{2 \times 1}{3\frac{\pi}{2}} R = \frac{4R}{3\pi}}$$

$$\boxed{y_K = \frac{2\left(\cos 0 - \cos\frac{\pi}{2}\right)}{3\left(\frac{\pi}{2} - 0\right)} R = \frac{2R}{3\frac{\pi}{2}} = \frac{4R}{3\pi}}$$

Στην τελευταία περίπτωση του τεταρτοκυκλίου, δίνονται στη συνέχεια 2 ακόμη τρόποι.

4-23

(α) $\quad (x^2+y^2=R^2)$



$$dA = y\,dx = \sqrt{R^2-x^2}\,dx$$

$$A=\int_A dA=\int_0^R \sqrt{R^2-x^2}\,dx =$$

$$=\int_0^R \frac{R^2-x^2}{\sqrt{R^2-x^2}}\,dx = R^2\int_0^R \frac{dx}{\sqrt{R^2-x^2}}-\int_0^R\frac{x^2dx}{\sqrt{R^2-x^2}} \quad (1)$$

$$\int_0^R \frac{dx}{\sqrt{R^2-x^2}} \overset{\text{γνωστό ολοκλ.}}{=\!=} \left(\arcsin\frac{x}{R}\right)_0^R = \arcsin 1-\arcsin 0 = \frac{\pi}{2}-0 \quad (1.\alpha)$$

$$\int_0^R \frac{x^2dx}{\sqrt{R^2-x^2}} = \int_0^R x\,\frac{x\,dx}{\sqrt{R^2-x^2}}\ ,\quad x\equiv u\ :\ dx=du$$

$$dv\equiv \frac{x\,dx}{\sqrt{R^2-x^2}}\ :\int:\ v=\int\frac{x\,dx}{\sqrt{R^2-x^2}} \overset{\text{γνωστό}}{=\!=} -\sqrt{R^2-x^2}$$

Ολοκλήρωση κατά παράγοντες : $\displaystyle\int_0^R u\,dv=[u\cdot v]_0^R-\int_0^R v\,du$

$$\int_0^R \frac{x^2dx}{\sqrt{R^2-x^2}}=\int_0^R \underset{u}{x}\,\underbrace{\frac{x\,dx}{\sqrt{R^2-x^2}}}_{dv} = \left[x\cdot\left(-\sqrt{R^2-x^2}\right)\right]_0^R-\int_0^R -\sqrt{R^2-x^2}\,dx =$$

$$=-\left(R\sqrt{R^2-R^2}-0\cdot\sqrt{R^2-0}\right)+\underbrace{\int_0^R\sqrt{R^2-x^2}\,dx}_{=A} = A \quad (1.\beta)$$

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(1), (1.α), (1.β) $\Rightarrow$

$$A = R^2\frac{\pi}{2}-A \Rightarrow 2A=\frac{\pi R^2}{2} \Rightarrow \boxed{A=\frac{\pi R^2}{4}} \quad \text{(αναμενόμενο)}$$

(4-24)

$$S_y=\int_A x\,dA = \int_0^R x\sqrt{R^2-x^2}\,dx \qquad (2)$$

Ολοκλήρωση με αντ/ση : $\sqrt{R^2-x^2}\equiv t$

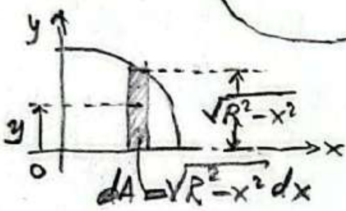
$$x=\sqrt{R^2-t^2} \;:\; \frac{dx}{dt}=\frac{1}{2\sqrt{R-t^2}}(-2t) \Rightarrow dx=\frac{-t\,dt}{\sqrt{R^2-t^2}}$$

$$\Rightarrow S_y=\int\sqrt{R^2-t^2}\cdot t\cdot\frac{-t\,dt}{\sqrt{R^2-t^2}}=-\int t^2dt=-\frac{t^3}{3}=$$

$$=-\frac{1}{3}\left(R^2-x^2\right)^{\frac{3}{2}}\Big|_0^R=-\frac{1}{3}\left[\left(R^2-R^2\right)^{\frac{3}{2}}-\left(R^2-0\right)^{\frac{3}{2}}\right]=\boxed{\frac{R^3}{3}}$$

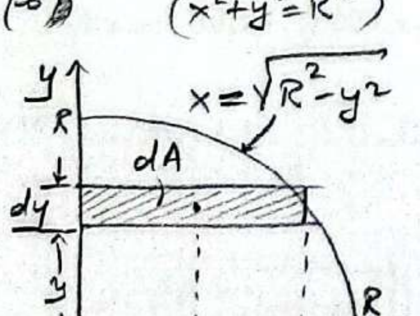
$$\boxed{S_x}=\int_A y\,dA=\int_0^R\frac{\sqrt{R^2-x^2}}{2}\sqrt{R^2-x^2}\,dx=\frac{1}{2}\int_0^R\left(R^2-x^2\right)dx=$$

$$=\frac{1}{2}\left(R^2\cdot R-\frac{x^3}{3}\Big|_0^R\right)=\frac{1}{2}\,\frac{3R^3-R^3}{3}=\boxed{\frac{R^3}{3}}$$



$$\boxed{x_K=\frac{\int_A x\,dA}{\int_A dA}=\frac{S_y}{A}=\frac{R^3}{3}\cdot\frac{4}{\pi R^2}=\frac{4R}{3\pi}=y_K}$$

(β) $(x^2+y^2=R^2)$ $\qquad dA=\sqrt{R^2-y^2}\,dy$



$$\boxed{A}=\int_A dA=\int_0^R\sqrt{R^2-y^2}\,dy=\int_0^R\frac{R^2-y^2}{\sqrt{R^2-y^2}}\,dy$$

$$=R^2\underbrace{\int_0^R\frac{dy}{\sqrt{R^2-y^2}}}_{\text{γνωστό ολ.}}-\int_0^R \underbrace{y}_{u}\cdot\underbrace{\frac{y}{\sqrt{R^2-y^2}}\,dy}_{dv}=\ldots\;\boxed{\frac{\pi R^2}{4}}$$

dv ... ολοκλήρ. κατά παράγ.

$$\boxed{S_y=\int_A x\,dA=\int_0^R\frac{\sqrt{R^2-y^2}}{2}\sqrt{R^2-y^2}\,dy=\frac{1}{2}\int_0^R\left(R^2-y^2\right)dy=\ldots=\frac{R^3}{3}}$$

...