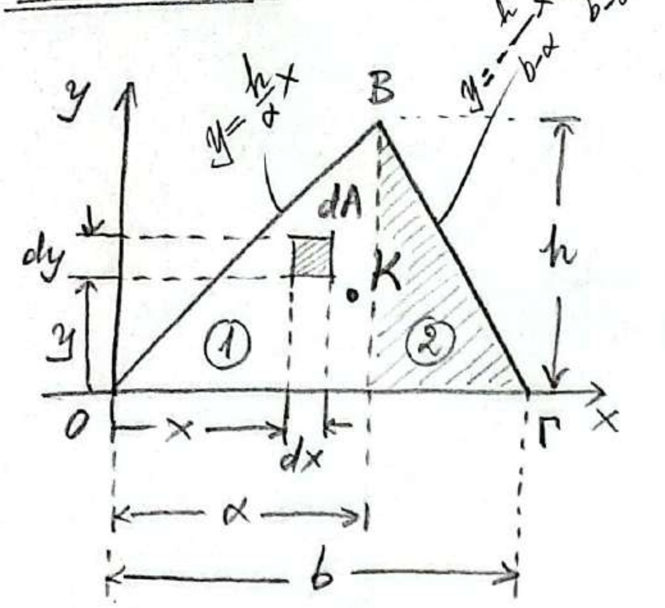
(4-13)

23η ΑΣΚΗΣΗ



Να υπολογιστούν οι συντεταγμένες x_K, y_K του κεντροειδούς K της <u>τριγωνικής</u> <u>επιφάνειας</u>

<u>Λύση</u>

$$\left.\begin{array}{l} \underline{OB}:\ y=\gamma' x+\delta \\ x=0,\ y=0 \Rightarrow \delta=0 \\ x=\alpha,\ y=h \Rightarrow \gamma'=\frac{h}{\alpha} \end{array}\right\} \Rightarrow y=\frac{h}{\alpha}x$$

$$\underline{B\Gamma}:\ y=\gamma' x+\delta$$

$$\left.\begin{array}{l} x=\alpha,\ y=h \Rightarrow \delta=h-\gamma'\alpha \\ x=b,\ y=0 \Rightarrow \delta=-\gamma' b \end{array}\right\} \Rightarrow \left.\begin{array}{l} \gamma'=-\frac{h}{b-\alpha} \\ \delta=\frac{bh}{b-\alpha} \end{array}\right\} \Rightarrow y=-\frac{h}{b-\alpha}x+\frac{bh}{b-\alpha}$$

<u>1ος τρόπος</u>

$$S_y=\int_A x\,dA=\int_{A_1} x\,dA+\int_{A_2} x\,dA=\int_{x=0}^{x=\alpha} x\left(\int_{y=0}^{y=\frac{h}{\alpha}x} dy\right)dx+\int_{x=\alpha}^{x=b} x\left(\int_{y=0}^{y=-\frac{h}{b-\alpha}x+\frac{bh}{b-\alpha}} dy\right)dx=$$

$$=\int_0^{\alpha} x\,\frac{h}{\alpha}x\,dx+\int_{\alpha}^{b} x\left(-\frac{h}{b-\alpha}x+\frac{bh}{b-\alpha}\right)dx=$$

$$=\frac{h}{\alpha}\frac{x^3}{3}\Big|_0^{\alpha}-\frac{h}{b-\alpha}\frac{x^3}{3}\Big|_{\alpha}^{b}+\frac{bh}{b-\alpha}\frac{x^2}{2}\Big|_{\alpha}^{b}=\frac{h\alpha^2}{3}-\frac{h}{b-\alpha}\left(\frac{b^3-\alpha^3}{3}-b\frac{b^2-\alpha^2}{2}\right)=$$

$$=\frac{h\alpha^2}{3}-\frac{h}{b-\alpha}\frac{\cancel{2b^3}-2\alpha^3-\cancel{3}b^3+3\alpha^2 b}{6}=\frac{h\alpha^2}{3}-\frac{h}{\cancel{b-\alpha}}\frac{2\alpha^2\cancel{(b-\alpha)}-b\cancel{(b^2-\alpha^2)}(b+\alpha)}{6}=$$

$$=\cancel{\frac{h\alpha^2}{3}}-\frac{h}{6}\left(\cancel{2\alpha^2}-b^2-\alpha b\right)=\boxed{\frac{bh}{6}(\alpha+b)}$$

$$A=\int_A dA=\int_{A_1} dA+\int_{A_2} dA=\int_0^{\alpha}\left(\int_0^{\frac{h}{\alpha}x} dy\right)dx+\int_{\alpha}^{b}\left(\int_0^{-\frac{h}{b-\alpha}x+\frac{bh}{b-\alpha}} dy\right)dx=$$

$$=\frac{h}{\alpha}\frac{x^2}{2}\Big|_0^{\alpha}-\frac{h}{b-\alpha}\frac{x^2}{2}\Big|_{\alpha}^{b}+\frac{bh}{b-\alpha}x\Big|_{\alpha}^{b}=\frac{h\alpha}{2}-\frac{h}{\cancel{b-\alpha}}\left(\frac{\cancel{b^2-\alpha^2}^{\,b+\alpha}}{2}-b\cancel{(b-\alpha)}\right)=$$

$$=\cancel{\frac{h\alpha}{2}}-\frac{h}{2}(\cancel{\alpha}-b)=\boxed{\frac{bh}{2}}$$ (προφανώς αναμενόμενο)

$$\boxed{x_K = \frac{\int_A x\,dA}{\int_A dA} = \frac{S_y}{A} = \frac{\cancel{bh}}{\cancel{6}_{3}}(\alpha+b)\cdot\frac{2}{\cancel{bh}} = \frac{\alpha+b}{3}}$$

$$S_x = \int_A y\,dA = \int_{A_1} y\,dA + \int_{A_2} y\,dA = \int_{x=0}^{x=\alpha}\left(\int_{y=0}^{y=\frac{h}{\alpha}x} y\,dy\right)dx + \int_{x=\alpha}^{x=b}\left(\int_{y=0}^{y=-\frac{h}{b-\alpha}x+\frac{bh}{b-\alpha}} y\,dy\right)dx =$$

$$= \int_0^{\alpha}\left(\frac{y^2}{2}\right)_0^{\frac{h}{\alpha}x}dx + \int_{\alpha}^{b}\left(\frac{y^2}{2}\right)_0^{-\frac{h}{b-\alpha}x+\frac{bh}{b-\alpha}}dx = \frac{1}{2}\int_0^{\alpha}\frac{h^2}{\alpha^2}x^2\,dx + \frac{1}{2}\int_{\alpha}^{b}\left(-\frac{h}{b-\alpha}x+\frac{bh}{b-\alpha}\right)^2dx =$$

$$= \frac{h^2}{2\alpha^2}\frac{\alpha^3}{3} + \frac{h^2}{2(b-\alpha)^2}\int_{\alpha}^{b}\left(x^2+b^2-2bx\right)dx = \frac{\alpha h^2}{6} + \frac{h^2}{2(b-\alpha)^2}\left(\frac{b^3-\alpha^3}{3} + b^2(b-\alpha) - 2b\frac{b^2-\alpha^2}{2}\right) =$$

$$= \frac{\alpha h^2}{6} + \frac{h^2}{2(b-\alpha)^{\cancel{2}}}\left[\frac{\cancel{(b-\alpha)}(b^2+\alpha b+\alpha^2)}{3} + b^2\cancel{(b-\alpha)} - b\cancel{(b-\alpha)}(b+\alpha)\right] =$$

$$= \frac{\alpha h^2}{6} + \frac{h^2}{2(b-\alpha)}\;\frac{b^2\cancel{+\alpha b}+\alpha^2\cancel{+3b^2}\cancel{-3b^2}-\cancel{3}^{2}\alpha b}{3} = \frac{\cancel{\alpha h^2}}{\cancel{6}} + \frac{h^2}{6\cancel{(b-\alpha)}}(b-\cancel{\alpha})^{\cancel{2}} =$$

$$\boxed{= \frac{bh^2}{6}}$$

$$\boxed{y_K = \frac{\int_A y\,dA}{\int_A dA} = \frac{S_x}{A} = \frac{\cancel{b}h^{\cancel{2}}}{6}\frac{2}{\cancel{bh}} = \frac{h}{3}}$$

$2^{ος}$ τρόπος

$x_{K_1} = \frac{2}{3}\alpha,\quad y_{K_1} = \frac{1}{3}h,\quad A_1 = \frac{\alpha h}{2}$ (Βαρύκεντρο στο $\frac{1}{3}$ του ύψους)

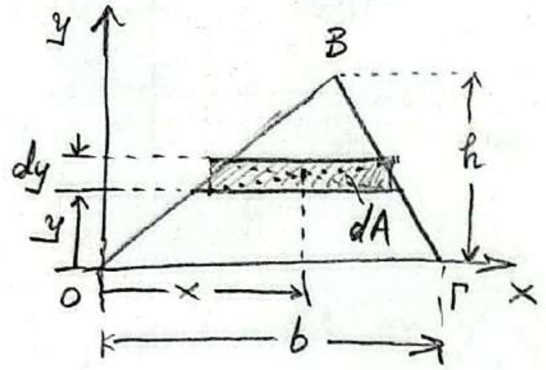
$x_{K_2} = \alpha + \frac{b-\alpha}{3} = \frac{b+2\alpha}{3},\quad y_{K_2} = \frac{1}{3}h,\quad A_2 = \frac{h}{2}(b-\alpha)$

$$\boxed{x_K = \frac{\Sigma x_{K_i}A_i}{\Sigma A_i} = \frac{\frac{2}{3}\alpha\left(\frac{\alpha h}{\cancel{2}}\right) + \frac{b+2\alpha}{3}\left(\frac{h}{\cancel{2}}(b-\alpha)\right)}{\left(\frac{\alpha h}{\cancel{2}}\right) + \left(\frac{h}{\cancel{2}}(b-\alpha)\right)} =}$$

$$= \frac{\cancel{2\alpha^2 h} + hb^2 \cancel{- hb\alpha} + 2h\alpha b - \cancel{2h\alpha^2}}{\cancel{3\alpha h} + 3bh - \cancel{3\alpha h}} = \frac{\cancel{h}b(b+\alpha)}{3\cancel{bh}} = \boxed{\frac{b+\alpha}{3}}$$

$$\boxed{y_{K_1} = \frac{\Sigma y_{K_i} A_i}{\Sigma A_i} = \frac{\frac{1}{3}h\left(\cancel{\frac{\alpha h}{2}}\right) + \frac{1}{3}h\left(\frac{h}{2}(b-\cancel{\alpha})\right)}{\left(\cancel{\frac{\alpha h}{2}}\right) + \left(\frac{h}{2}(b-\cancel{\alpha})\right)} = \frac{\frac{h^2 b}{6}}{\frac{hb}{2}} = \frac{h}{3}}$$

$3^{ος}$ τρόπος



$\underline{OB}$: $y = \frac{h}{\alpha}x \Rightarrow x = \frac{\alpha}{h}y$

$\underline{B\Gamma}$: $y = -\frac{h}{b-\alpha}x + \frac{bh}{b-\alpha} \Rightarrow x = b - \frac{b-\alpha}{h}y$

$$dA = \left(b - \frac{b-\alpha}{h}y - \frac{\alpha}{h}y\right)dy = \left(b - \frac{b}{h}y\right)dy \quad (*)$$

$$x = \frac{\frac{\alpha}{h}y + b - \frac{b-\alpha}{h}y}{2} = \frac{1}{2}\left(b - \frac{b-2\alpha}{h}y\right)$$

$$S_y = \int_A x\,dA = \int_{y=0}^{y=h} \frac{1}{2}\left(b - \frac{b-2\alpha}{h}y\right)\left(b - \frac{b}{h}y\right)dy =$$

$$= \frac{1}{2}\int_0^h \left(b^2 - \frac{b^2}{h}y - b\frac{b-2\alpha}{h}y + b\frac{b-2\alpha}{h^2}y^2\right)dy =$$

$$= \frac{1}{2}\left(b^2 y\Big|_0^h - \frac{b^2}{h}\frac{y^2}{2}\Big|_0^h - b\frac{b-2\alpha}{h}\frac{y^2}{2}\Big|_0^h + b\frac{b-2\alpha}{h^2}\frac{y^3}{3}\Big|_0^h\right) =$$

$$= \frac{b}{2}\left(bh - \frac{b}{2\cancel{h}}h^{\cancel{2}} - \frac{b-2\alpha}{2\cancel{h}}h^{\cancel{2}} + \frac{b-2\alpha}{3\cancel{h^2}}h^{\cancel{3}}\right) =$$

$$= \frac{b}{2}\left(\cancel{bh} - \cancel{\frac{bh}{2}} - \cancel{\frac{bh}{2}} + \alpha h + \frac{bh}{3} - \frac{2\alpha h}{3}\right) = \boxed{\frac{bh}{6}(b+\alpha)} \checkmark$$

$$S_x = \int_A y\,dA = \int_{y=0}^{y=h} y\left(b - \frac{b}{h}y\right)dy = b\left(\frac{y^2}{2}\Big|_0^h - \frac{1}{h}\frac{y^3}{3}\Big|_0^h\right) =$$

$$= b\left(\frac{h^2}{2} - \frac{1}{\cancel{h}}\frac{h^{\cancel{3}\,2}}{3}\right) = \boxed{\frac{bh^2}{6}} \checkmark \ldots$$