Functional Analysis I
(Solutions of problem sheet 3)

Exercise 1. *Let X be a normed space.*

(i) A subset C of X is called convex if for all $x, y \in C$ and $t \in [0,1]$

$$tx + (1-t)y \in C\,.$$

Show that $B(0,1)$ is convex.

(ii) A convex subset C of X is called strictly convex if for all $x, y \in C$ and $t \in (0,1)$

$$tx + (1-t)y \in C^{\circ}\,.$$

Is the closed unit ball of $(\mathbb{R}^2, \|\cdot\|_\infty)$ strictly convex?

Solution.

(i) If $x, y \in B(0,1)$ and $t \in [0,1]$, then

$$\|tx + (1-t)\| \leq t\|x\| + (1-t)\|y\| \leq t + 1 - t = 1$$

and hence $B(0,1)$ is convex.

(ii) The closed unit ball of $(\mathbb{R}^2, \|\cdot\|_\infty)$ is not strictly convex. Indeed if $x = (1,0)\, y = (1,1)$, then $\|x\| = \|y\| = 1$ but

$$\|\frac{x+y}{2}\|_\infty = \|(1, \frac{1}{2})\|_\infty = 1$$

and thus $\frac{x+y}{2}$ does not belong to its interior.

□

Exercise 2. *Let $g : [0,1] \to \mathbb{R}$ with $g(x) > 0$, for all $x \in [0,1]$. If $f \in C[0,1]$ we define*

$$\|f\|_g = \sup_{x \in [0,1]} |f(x)| g(x)\,.$$

(i) Show that $\|\cdot\|_g$ is a norm on $C[0,1]$.

(ii) If $\inf_{x\in[0,1]} g(x) = m > 0$ and $\sup_{x\in[0,1]} g(x) =< +\infty$, show that $\|\cdot\|_g$ is an equivalent to $\|\cdot\|_\infty$.

Solution.

(i) Obviously $\|f\|_g \geq 0$, for all $f \in C[0,1]$. Moreover if $\|f\|_g = 0$, then $\sup_{x\in[0,1]} |f(x)|g(x) = 0$ and hence $|f(x)|g(x) = 0$, for all $x \in [0,1]$ which by the hypothesis $g(x) > 0$ gives $|f(x)| = 0$, for all $x \in [0,1]$, i.e. $f = 0$. $\|\lambda f\|_g = \sup_{x\in[0,1]} |\lambda f(x)|g(x) = |\lambda| \sup_{x\in[0,1]} |f(x)|g(x) = |\lambda|\|f\|_g$, for all

$\lambda \in \mathbb{R}$ and all $f \in C[0,1]$.
For all $f_1, f_2 \in C[0,1]$ we have that

$$\begin{aligned}\|f_1 + f_2\|_g &\leq \sup_{x\in[0,1]} (|f_1(x)| + |f_2(x)|)g(x) \\ &\leq \sup_{x\in[0,1]} |f_1(x)|g(x) + \sup_{x\in[0,1]} |f_2(x)|g(x) \\ &= \|f_1\|_g + \|f_2\|_g .\end{aligned}$$

(ii) We have that $m\|f\|_g \leq \|f\|_\infty \leq M\|f\|_g$, $f \in C[0,1]$. Hence the two norms are equivalent.

□

Exercise 3. *(i) Show that the unit sphere*

$$S = \{x \in X : \|x\| = 1\}$$

of a normed space is closed.

(ii) Show that the unit sphere of l^2 is not compact.

Solution.

(i) The result follows immediately from the continuity of the norm and the fact that S is the inverse image of $\{1\}$.

(ii) Consider the sequence (e_n) where $e_n = (0, ..., 0, 1, 0, ...)$ with 1 in the n-th position. Then for all $n \neq m$ we have that $\|e_n\| = \|e_m\| = 1$ but $\|e_n - e_m\| = 1$. Hence (e_n) does not have any Cauchy subsequence.

This is characteristic of infinite dimensional spaces as it is well known that in finite dimensions every closed and bounded set is compact. □

Exercise 4. *Let X be a normed space and Y a subspace of X with $Y \neq X$. Show that the set Y^c is dense in X.*

Solution. Since $X = Y \cup Y^c$, it is enough to show that for all $x \in Y$ and $\varepsilon > 0$ there exists $z \in Y^c$, such that $z \in B(x, \varepsilon)$. Since $Y \neq X$, using the Theorem that we proved in class Y has empty interior, hence

$$B(x, \varepsilon) \cap Y^c \neq \emptyset$$

and the result follows. □

Exercise 5. *Show that c_{00} cannot be a Banach space.*

Solution. The elements $e_n = (0, ..., 0, 1, 0, ...)$ with 1 in the n-th position, $n = 1, 2, 3, ...$, are a countable basis of c_{00} and hence the latter cannot be a Banach space. □

Exercise 6. *Let* $X = C^1([-1,1])$*. Set* $\|f\|_1 = \|f\|_\infty$*,* $\|f\|_2 = \|f'\|_\infty$ *and* $\|f\| = \|f\|_1 + \|f\|_2$*. Show that*

(i) $(X, \|\cdot\|_1)$*is a normed space but not a Banach space,*

(ii) $\|\cdot\|_2$ *is not a norm on* X*,*

(iii) $(X, \|\cdot\|)$ *is a Banach space.*

Solution.

(i) If

$$f_n(x) = (x^2 + \frac{1}{n})^{\frac{1}{2}}\,, n = 1, 2, ... \text{ and } f(x) = |x|, \text{ for all } x \in [-1,1]\,,$$

then f_n converges uniformly to f, i.e. $\|f_n - f\|_1 = \|f_n - f\|_\infty \to 0$, as $n \to +\infty$. But $f \notin X$ and hence X is not a Banach space.

(ii) If $f(x) = c$, for all $x \in [-1,1]$, then $\|f\|_2 = \|f'\|_\infty = 0$ while $f \neq 0$. Hence $\|\cdot\|_2$ is not a norm on X,

(iii) Let (f_n) a Cauchy sequence with respect to $\|\cdot\|$. Then (f_n) and (f_n') are Cauchy sequences with respect to $\|\cdot\|_\infty$ and hence $f_n \to f$ and $f_n' \to g$ in $C([-1,1])$. To conclude the proof it is enough to show that f is differentible with $f' = g$.

To ths end let $\varepsilon > 0$ and $t \in [-1,1]$. Since g is continuous in $[-1,1]$ it is uniformly continuous and thus there exists $\delta > 0$ such that

$$|g(t+s) - g(t)| < \varepsilon\,, \text{ for all } |s| < \delta\,.$$

If $h \in \mathbb{R}$ with $0 < |h| < \delta$, then

$$\begin{aligned} |\frac{f(t+h) - f(t)}{h} - g(t)| &= \lim |\frac{f_n(t+h) - f_n(t)}{h} - g(t)| \\ &= \lim |\frac{1}{h}\int_0^h f_n'(t+s)\,ds - g(t)| \\ &= |\frac{1}{h}\int_0^h (g(t+s) - g(t))\,ds| \\ &\leq \frac{1}{|h|}\varepsilon|h| = \varepsilon\,. \end{aligned}$$

Hence f is differentiable with $f' = g$. Therefore $(X, \|\cdot\|)$ is a Banach space.

□