Functional Analysis I
(Problem sheet 4)

Exercise 1. *Let $T \in B(X,Y)$. Show that*

(i)

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| \leq 1\} \\ &= \sup\{\|Tx\| : \|x\| = 1\}\,.\end{aligned}$$

(ii)

$$\|T\| = \inf\{M > 0 : \|Tx\| \leq M\|x\|\,,\ \text{for all } x \in X\}\,.$$

Exercise 2 (Multiplication operator)**.** *Consider $X = C[a,b]$ equipped with $\|\cdot\|_\infty$ and $g \in X$. Define $T : X \to X$ by*

$$Tf(x) = g(x)f(x)\,,\ \text{for all } x \in [a,b]\,.$$

Show that $T \in B(X)$.

Exercise 3 (Right shift operator)**.** *Let $S : l^1 \to l^1$ defined by*

$$S(x_1, x_2, ...) = (0, x_1, x_2, ...)\,.$$

Show that S is bounded and find its norm.

Exercise 4. *Show that if X is a finite dimensional normed space, then every operator $T : X \to Y$, where Y is a normed space, is bounded.*

Exercise 5. *Let X be a Banach space.*

(i) Suppose that $T \in B(X)$ and for all $y \in X$, the series

$$\sum_{n=1}^{\infty} \|T^n y\|$$

converges. Show that for all $y \in X$, the equation

$$x = y + Tx$$

has a unique solution.

(ii) We suppose that $T \in B(X)$ and $\|T\| < 1$. Show that the operator $I - T$ is "1-1" and onto.

(iii) Let $T : l^\infty \to l^\infty$ defined by

$$T(x_1, x_2, x_3, ...) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, ...)$$

Show that for all $y \in l^\infty$, the equation

$$x = y + Tx$$

has a unique solution.