Functional Analysis I
(Problem sheet 3)

Exercise 1. *Let X be a normed space.*

(i) A subset C of X is called convex if for all $x, y \in C$ and $t \in [0,1]$

$$tx + (1-t)y \in C\,.$$

Show that $B(0,1)$ is convex.

(ii) A convex subset C of X is called strictly convex if for all $x, y \in C$ and $t \in (0,1)$

$$tx + (1-t)y \in C^{\circ}\,.$$

Is the closed unit ball of $(\mathbb{R}^2, \|\cdot\|_\infty)$ strictly convex?

Exercise 2. *Let $g : [0,1] \to \mathbb{R}$ with $g(x) > 0$, for all $x \in [0,1]$. If $f \in C[0,1]$ we define*

$$\|f\|_g = \sup_{x\in[0,1]} |f(x)|g(x)\,.$$

(i) Show that $\|\cdot\|_g$ is a norm on $C[0,1]$.

(ii) If $\inf_{x\in[0,1]} g(x) = m > 0$ and $\sup_{x\in[0,1]} g(x) =< +\infty$, show that $\|\cdot\|_g$ is an equivalent to $\|\cdot\|_\infty$.

Exercise 3. *(i) Show that the unit sphere*

$$S = \{x \in X : \|x\| = 1\}$$

of a normed space is closed.

(ii) Show that the unit sphere of l^2 is not compact.

Exercise 4. *Let X be a normed space and Y a subspace of X with $Y \neq X$. Show that the set Y^c is dense in X.*

Exercise 5. *Show that c_{00} cannot be a Banach space.*

Exercise 6. *Let $X = C^1([-1,1])$. Set $\|f\|_1 = \|f\|_\infty$, $\|f\|_2 = \|f'\|_\infty$ and $\|f\| = \|f\|_1 + \|f\|_2$. Show that*

(i) $(X, \|\cdot\|_1)$is a normed space but not a Banach space,

(ii) $\|\cdot\|_2$ is not a norm on X,

(iii) $(X, \|\cdot\|)$ is a Banach space.