Exercises in Functional Analysis
(problem sheet 2)

Exercise 1. *Let X be a normed space and Y a subspace of X. Show that the closure $\overline{Y}$ of Y is a subspace of X.*

Exercise 2. *Let Y a subspace of the Banach space $(X, \|\cdot\|)$. Show that $(Y, \|\cdot\|)$ is a Banach if and only if Y is closed.*

Exercise 3. *With l^∞ we denote the space of bounded real sequences. Show that l^∞ with the norm*

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \,, for\, x = (x_n) \in l^\infty$$

is a Banach space.

Exercise 4. *With c_0 we denote the space of real sequences coverging to zero. Show that c_0 with the norm $\|\cdot\|_\infty$ is a Banach space.*
(Hint: use exercise 2.)

Exercise 5. *Show that*

$$X = \{f \in C[a, b] : f(a) = 0\}$$

with the norm $\|\cdot\|_\infty$ is a Banach space.

Exercise 6. *We equip $C[a, b]$ with the norm*

$$\|f\|_2 = (\int_a^b f(x)^2\, dx)^{\frac{1}{2}} \,.$$

(i) Show that $\|\cdot\|_2$ is a norm.

(ii) Show that if the sequence (f_n) converges to f with respect to $\|\cdot\|_\infty$, then it converges to f with respect to $\|\cdot\|_2$.

(iii) Is $C[a, b]$ with $\|\cdot\|_2$ a Banach space?