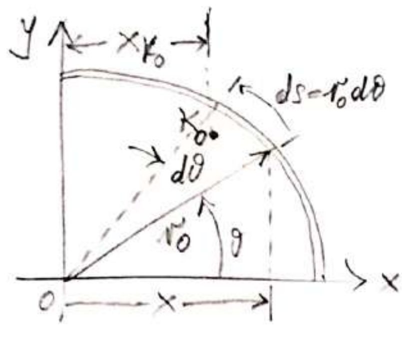
Κεντροειδές τεταρτοκυκλικού τόξου (Χρησιμοποιείται και στον 5ο τρόπο υπολογισμού του κεντροειδούς του τεταρτοκυκλίου)



$$x_{K_0} = \frac{\int x\,ds}{\int ds}, \quad \int ds = r_0 \int_0^{\frac{\pi}{2}} d\theta = \frac{r_0 \pi}{2}$$

$$\int x\,ds = \int_0^{\frac{\pi}{2}} r_0 \cos\theta \, r_0\, d\theta = r_0^2 \sin\theta \Big|_0^{\frac{\pi}{2}} = r_0^2$$

$$\boxed{x_{K_0} = \frac{r_0^2}{\frac{r_0 \pi}{2}} = \frac{2 r_0}{\pi}} \quad (*)$$

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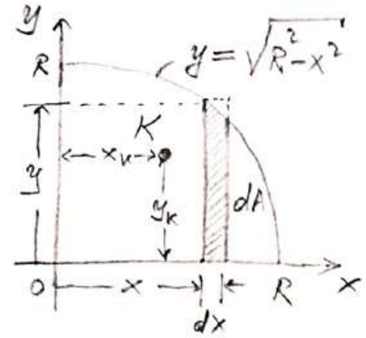
3) Κεντροειδές Τεταρτοκυκλίου

1ος Τρόπος



$$x_K = \frac{\int_A x\,dA}{\int_A dA}, \quad dA = y\,dx = \sqrt{R^2 - x^2}\,dx$$

Παρανόηση (εξήγηση), είναι ήδη: $A = \frac{\pi R^2}{4}$:

$$\int_A dA = \int_{x=0}^{x=R} \sqrt{R^2 - x^2}\,dx = \int_0^R \frac{R^2 - x^2}{\sqrt{R^2 - x^2}}\,dx =$$

$$= R^2 \int_0^R \frac{dx}{\sqrt{R^2 - x^2}} - \int_0^R \frac{x^2 dx}{\sqrt{R^2 - x^2}} \quad (3.1)$$

$$\boxed{\int_0^R \frac{dx}{\sqrt{R^2 - x^2}} \overset{\text{γνωστό ολοκλ.}}{=} \left(\arcsin \frac{x}{R}\right)\Big|_0^R} = \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 \quad (3.1\alpha)$$

ποιά γωνία έχει ημ = 1
ποιά γωνία έχει ημ = 0

$$\boxed{\int_0^R \frac{x^2 dx}{\sqrt{R^2 - x^2}}} = \int_0^R \underbrace{x}_{u} \underbrace{\frac{x\,dx}{\sqrt{R^2 - x^2}}}_{dv} =$$

$$\boxed{\begin{aligned} &x = u, \quad dx = du \quad (\text{ολοκλήρωση κατά παράγοντες}) \\ &dv = \frac{x\,dx}{\sqrt{R^2 - x^2}} \Rightarrow \frac{dv}{dx} = \frac{x}{\sqrt{R^2 - x^2}} \Rightarrow v = -\sqrt{R^2 - x^2} \end{aligned}}$$

$$= \Big[\underbrace{-\sqrt{R^2 - x^2}}_{v} \cdot \underbrace{x}_{u}\Big]_0^R - \int_0^R \underbrace{-\sqrt{R^2 - x^2}}_{v}\, \underbrace{dx}_{du} = 0 + \int_0^R \sqrt{R^2 - x^2}\,dx \quad (3.1\beta)$$

$$(3.1), (3.1\alpha), (3.1\beta) \Rightarrow \int_A dA = \int_0^R \sqrt{R^2 - x^2}\,dx = R^2 \frac{\pi}{2} - \int_0^R \sqrt{R^2 - x^2}\,dx \Rightarrow$$

$$\Rightarrow 2 \int_0^R \sqrt{R^2 - x^2}\,dx = R^2 \frac{\pi}{2} \Rightarrow \int_0^R \sqrt{R^2 - x^2}\,dx = A = \frac{\pi R^2}{4} \quad ✓$$

Αριθμητικά

$$\int_A x\,dA = \int_0^R x\sqrt{R^2-x^2}\,dx \quad (3.2)$$

Αντικατάσταση: $\sqrt{R^2-x^2} \equiv t \Rightarrow t^2 = R^2-x^2 \Rightarrow x=\sqrt{R^2-t^2}$

$$\left.\begin{aligned} &\frac{dx}{dt} = \frac{1}{2}\frac{-2t}{\sqrt{R^2-t^2}} = \frac{-t}{\sqrt{R^2-t^2}} \Rightarrow dx = \frac{-t\,dt}{\sqrt{R^2-t^2}} \end{aligned}\right\} (3.2\alpha)$$

οδηγεί στην απλοποίηση των ριζών.

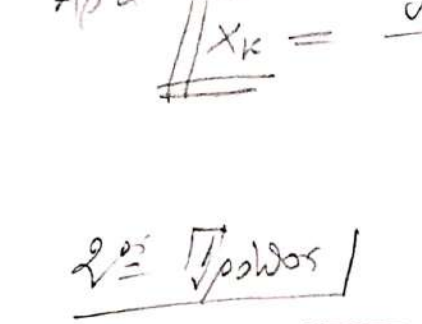
(3.2), (3.2α) ⇒

$$\int x\,dA = \int \cancel{\sqrt{R^2-t^2}}\cdot t\cdot\frac{-t\,dt}{\cancel{\sqrt{R^2-t^2}}} = -\int t^2\,dt = -\frac{t^3}{3} = \left[-\frac{1}{3}(R^2-x^2)^{3/2}\right]_0^R =$$

$$= +\frac{R^3}{3} \quad (3.3)$$

Άρα $\boxed{x_K = \frac{\int x\,dA}{A} = \frac{\frac{R^3}{3}}{\frac{\pi R^2}{4}} = \frac{4R}{3\pi}} \overset{\text{Προφανώς}}{=} \boxed{y_K}$ (3.4)

2ος Τρόπος



$$x_K = \frac{\int_A (\frac{x}{2})\,dA}{\int_A dA}, \quad x=\sqrt{R^2-y^2}, \quad dA = x\,dy = \sqrt{R^2-y^2}\,dy$$

Όσον αφορά στο $\int_A dA = \int_0^R \sqrt{R^2-y^2}\,dy =$

$$= \int_0^R \frac{R^2-y^2}{\sqrt{R^2-y^2}}\,dy = R^2\underbrace{\int_0^R\frac{dy}{\sqrt{R^2-y^2}}}_{\text{γνωστό ολοκλ.}} - \underbrace{\int_0^R \frac{y\,dy}{\sqrt{R^2-y^2}}\cdot y}_{\text{ολοκλ. κατά παράγοντες}} \ldots =$$

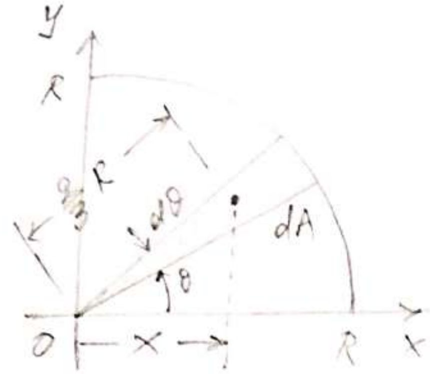
$$= \pi R^2/4$$

Όσον αφορά στο $\int_A (\frac{x}{2})\,dA = \int_{y=0}^{y=R}\frac{\sqrt{R^2-y^2}}{2}\sqrt{R^2-y^2}\,dy = \frac{1}{2}\int_0^R (R^2-y^2)\,dy =$

$$= \frac{R^2}{2}R - \frac{1}{2}\frac{y^3}{3}\Big|_0^R = \frac{R^3}{2} - \frac{R^3}{6} = \frac{R^3}{3} = (3.3) \checkmark$$

Άρα τελικά θα καταλήξουμε στην (3.4)

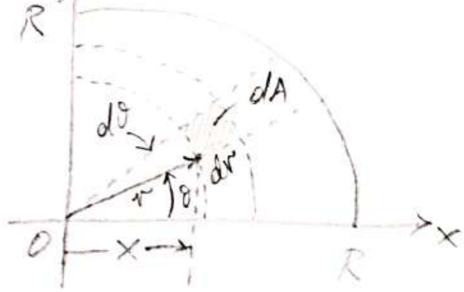
3ος Τρόπος



$$dA = \frac{R \cdot R\,d\theta}{2}, \quad x = \frac{2}{3}R\cos\theta$$

$$\boxed{x_K = \frac{\int_A x\,dA}{\int dA} = \frac{4}{\pi R^2}\int_0^{\pi/2}\frac{2R}{3}\cos\theta\,\frac{R^2 d\theta}{2} = \frac{4R}{3\pi}\sin\theta\Big|_0^{\frac{\pi}{2}} = \frac{4R}{3\pi}} = (3.4)$$

4ος Τρόπος



$$dA = dr \cdot r\,d\theta, \quad x = r\cos\theta$$

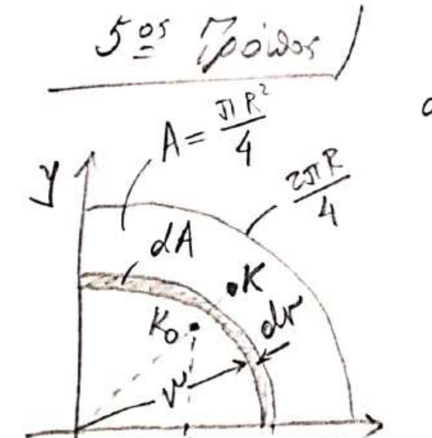
$$\int_A dA = \int_0^R r\left(\int_0^{\frac{\pi}{2}} d\theta\right) dr = \frac{r^2}{2}\Big|_0^R \cdot \theta\Big|_0^{\frac{\pi}{2}} = \frac{R^2\pi}{4} = A$$

$$\int_A x\,dA = \int_0^R r^2\left(\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\right) dr =$$

$$= \frac{r^3}{3}\Big|_0^R \sin\theta\Big|_0^{\frac{\pi}{2}} = \frac{R^3}{3}\cdot 1 = (3.3)$$

Άρα πάλι $\boxed{x_K = \int_A x\,dA \Big/ \int_A dA = \frac{4R}{3\pi}} = (3.4)$

5ος Τρόπος



$$dA = \frac{\pi r}{2}dr, \quad x \equiv x_{K_0} = \frac{2r}{\pi} \quad (*)$$

$$\int dA = \frac{\pi}{2}\int_0^R r\,dr = \frac{\pi}{2}\frac{r^2}{2}\Big|_0^R = \frac{\pi R^2}{4}$$

$$\int x\,dA = \int_0^R \frac{2r}{\pi}\,\frac{\pi r}{2}dr = \int_0^R r^2 dr = \frac{R^3}{3}$$

$$\boxed{x_K = \frac{\int x\,dA}{\int dA} = \frac{\frac{R^3}{3}}{\frac{\pi R^2}{4}} = \frac{4R}{3\pi}} = (3.4)$$