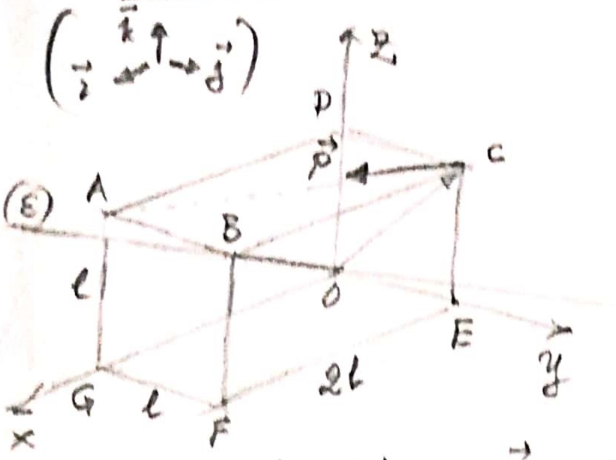
Άσκηση προς επίλυση 6. (Γεωργιάδης–Σιάσος, σελ. 79)



α) Ροπή $\vec{P}$ ως προς O

$$\vec{M}_0 = \vec{OC} \times \vec{P}$$

$$\vec{OC} = \ell\vec{j} + \ell\vec{k}$$

$$\vec{P} = P\,\vec{e}_{CA}\,, \quad \vec{e}_{CA} = \frac{\vec{CA}}{|\vec{CA}|}$$

$$\vec{CA} = 2\ell\vec{i} - \ell\vec{j}\,, \quad |\vec{CA}| = \sqrt{4\ell^2 + \ell^2} = \ell\sqrt{5}\,, \quad \vec{e}_{CA} = \frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}$$

$$\vec{P} = \frac{2P}{\sqrt{5}}\vec{i} - \frac{P}{\sqrt{5}}\vec{j}$$

$$\boxed{\vec{M}_0 = (\ell\vec{j} + \ell\vec{k}) \times \left(\frac{2P}{\sqrt{5}}\vec{i} - \frac{P}{\sqrt{5}}\vec{j}\right) = \frac{P\ell}{\sqrt{5}}\left(-2\vec{k} + 2\vec{j} + \vec{i}\right) \ \text{kN.m}}$$

β) Ροπή της $\vec{P}$ ως προς x-άξονα

$$\boxed{\vec{M}_{0x} = (\vec{M}_0 \cdot \vec{i})\,\vec{i} = \vec{i}\,\frac{P\ell}{\sqrt{5}}(\vec{i} + 2\vec{j} - 2\vec{k})\cdot\vec{i} = \frac{P\ell}{\sqrt{5}}\,\vec{i}} \quad \text{kNm}$$

γ) Ροπή της $\vec{P}$ ως προς OB

$$\vec{M}_{OB} = (\vec{M}_0 \cdot \vec{OB})\,\vec{e}_{OB}\,, \quad \vec{e}_{OB} = \frac{\vec{OB}}{|\vec{OB}|}\,, \quad \vec{OB} = 2\ell\vec{i} + \ell\vec{j} + \ell\vec{k}$$

$$|\vec{OB}| = \sqrt{4\ell^2 + 2\ell^2} = \ell\sqrt{6}$$

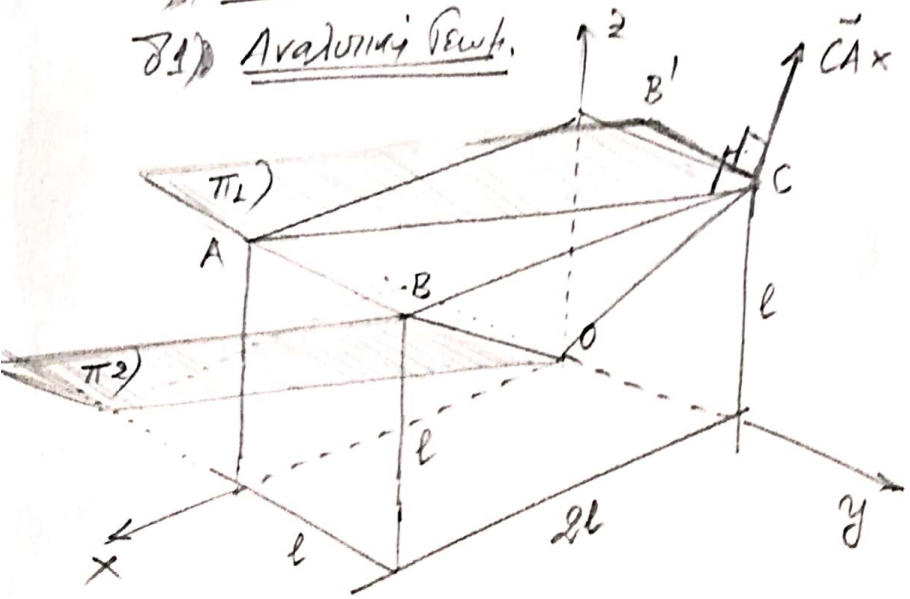
$$\vec{e}_{OB} = \frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}$$

$$\vec{M}_{OB} = \left[\frac{P\ell}{\sqrt{5}}\frac{1}{\sqrt{6}}(\vec{i} + 2\vec{j} - 2\vec{k})\cdot(2\vec{i} + \vec{j} + \vec{k})\right]\vec{e}_{OB} =$$

$$= \frac{P\ell}{\sqrt{30}}(2 + 2 - 2)\,\vec{e}_{OB} = \boxed{\frac{2P\ell}{\sqrt{30}}\,\vec{e}_{OB}} \quad \text{kNm}$$

δ) Κάθετη απόσταση των ευθειών CA και OB

δ1) Αναλυτική Γεωμ.



Η κοινή κάθετος των CA και OB, έχει τη διεύθυνση του διανύσματος

$$\vec{n} = \frac{\vec{CA}\times\vec{OB}}{|\vec{CA}\times\vec{OB}|}.$$

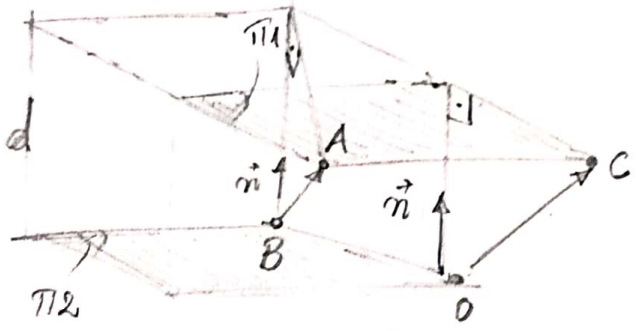
Το μήκος d της καθέτου θα είναι

$d = \vec{BA}\cdot\vec{n} = \vec{OC}\cdot\vec{n}$ (τα BA, OC ανήκουν τα επίπεδα π_1, π_2 των οποίων η κάθετη απόσταση είναι d)

$\vec{CA} = 2l\vec{i} - l\vec{j}$, $\vec{OB} = 2l\vec{i} + l\vec{j} + l\vec{k}$

$\vec{CA}\times\vec{OB} = 2l^2\vec{k} - 2l^2\vec{j} + 2l^2\vec{k} - l^2\vec{i} =$

$= l^2(-\vec{i} - 2\vec{j} + 4\vec{k})$



$|\vec{CA}\times\vec{OB}| = l^2\sqrt{21}$

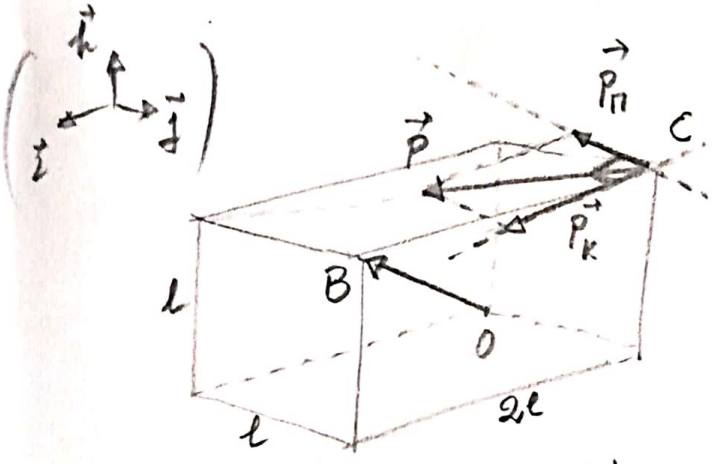
$\vec{n} = \frac{-\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{21}}$

(Βοηθητικό σχήμα)

$$\boxed{d = \vec{BA}\cdot\vec{n} = -l\vec{j}\cdot\frac{-\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{21}} = \frac{2l}{\sqrt{21}}\ m}$$

$$\text{ή } d = \vec{OC}\cdot\vec{n} = (l\vec{j} + l\vec{k})\cdot\frac{-\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{21}} = -\frac{2l}{\sqrt{21}} + \frac{4l}{\sqrt{21}} = \frac{2l}{\sqrt{21}}\ m \checkmark$$

δ2) Με τη βοήθεια του ροπού $|\vec{M}_{OB}|$



Βρήκαμε στο (γ) ότι,

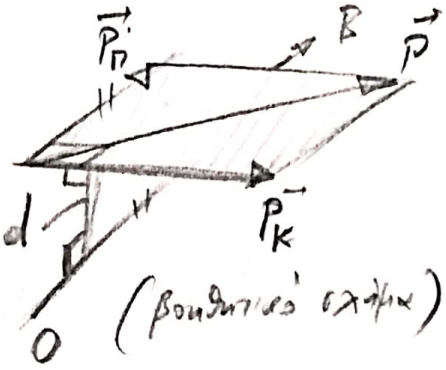
$$|\vec{M}_{OB}| = \frac{2Pl}{\sqrt{30}} \text{ kNm}$$

Αν αναλύσουμε την $\vec{P}$ ως:

$\vec{P} = \vec{P}_\pi + \vec{P}_k$, $\vec{P}_\pi \parallel \vec{OB}$, τότε

ροπή ως προς $\vec{OB}$ άξονα

προκαλεί μόνο η $\vec{P}_k$.



$$\vec{P}_\pi = (\vec{P}\cdot\vec{e}_{OB})\,\vec{e}_{OB}, \quad (\vec{P}, \vec{e}_{OB}, \text{σε } 1.5)$$

$$\vec{P}_\pi = \left(\frac{2P}{\sqrt{5}}\vec{i} - \frac{P}{\sqrt{5}}\vec{j}\right)\cdot\left(\frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\right)\left(\frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\right)$$

$$= \left(\frac{4P}{\sqrt{30}} - \frac{P}{\sqrt{30}}\right)(\cdots) =$$

$$= \frac{3P}{\sqrt{30}\sqrt{6}}(2\vec{i} + \vec{j} + \vec{k}) = \frac{3\sqrt{6}P}{\sqrt{30}\cdot 6_2}(\cdots) =$$

$$= \frac{\sqrt{6}P}{2\sqrt{5}\sqrt{6}}(\cdots) \Rightarrow \vec{P}_\pi = \frac{P}{2\sqrt{5}}(2\vec{i} + \vec{j} + \vec{k})$$

$$\vec{P}_k = \vec{P} - \vec{P}_\pi = \frac{2P}{\sqrt{5}}\vec{i} - \frac{P}{\sqrt{5}}\vec{j} - \frac{P}{\sqrt{5}}\vec{i} - \frac{P}{2\sqrt{5}}\vec{j} - \frac{P}{2\sqrt{5}}\vec{k} =$$

$$= \frac{P}{\sqrt{5}}\vec{i} - \frac{3P}{2\sqrt{5}}\vec{j} - \frac{P}{2\sqrt{5}}\vec{k} = \frac{P}{2\sqrt{5}}(2\vec{i} - 3\vec{j} - \vec{k})$$

$$|\vec{P}_k| = \frac{P}{2\sqrt{5}}\sqrt{4+9+1} = \frac{P\sqrt{2}\sqrt{7}}{2\sqrt{5}} = \frac{P\cdot 2\sqrt{7}}{2\sqrt{2}\sqrt{5}} = \frac{P\sqrt{7}}{\sqrt{10}}$$

Πρέπει $|\vec{M}_{OB}| = d\,|\vec{P}_k| \to d = \dfrac{|\vec{M}_{OB}|}{|\vec{P}_k|} = \dfrac{2Pl}{\sqrt{30}}\dfrac{\sqrt{10}}{P\sqrt{7}} \Rightarrow$

$$\Rightarrow d = \frac{2l\sqrt{10}}{\sqrt{10}\sqrt{21}} \Rightarrow \boxed{d = \frac{2l}{\sqrt{21}}} \checkmark$$