

Algorithmic Game Theory

Solution concepts in games

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Solution concepts

Choosing a strategy...

- Given a game, how should a player choose his strategy?
 - Recall: we assume each player knows the other players' preferences but not what the other players will choose
- The most fundamental question of game theory
 - Clearly, the answer is not always clear
- We will start with 2-player games

Prisoner's Dilemma: The Rational Outcome

- Let's revisit prisoner's dilemma
- Reasoning of pl. 1:
 - If pl. 2 does not confess, then I should confess
 - If pl. 2 confesses, then I should also confess

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

- Similarly for pl. 2
- **Expected outcome for rational players:** they will both confess, and they will go to jail for 3 years each
 - **Observation:** If they had both chosen not to confess, they would go to jail only for 1 year, each of them would have a strictly better utility

Dominant strategies

- Ideally, we would like a strategy that would provide the best possible outcome, regardless of what other players choose

- Definition: A strategy s_i of pl. 1 is *dominant* if

$$u_1(s_i, t_j) \geq u_1(s', t_j)$$

for every strategy $s' \in S^1$ and every strategy $t_j \in S^2$

- Similarly for pl. 2, a strategy t_j is dominant if

$$u_2(s_i, t_j) \geq u_2(s_i, t')$$

for every strategy $t' \in S^2$ and for every strategy $s_i \in S^1$

Dominant strategies

Even better:

- Definition: A strategy s_i of pl. 1 is *strictly dominant* if

$$u_1(s_i, t_j) > u_1(s', t_j)$$

for every strategy $s' \in S^1$ and every strategy $t_j \in S^2$

- Similarly for pl. 2
- In prisoner's dilemma, strategy D (confess) is strictly dominant

Observations:

- There may be more than one dominant strategies for a player, but then they should yield the same utility under all profiles
- Every player can have at most one strictly dominant strategy
- A strictly dominant strategy is also dominant

Existence of dominant strategies

- Few games possess dominant strategies
- It may be too much to ask for
- E.g. in the Bach-or-Stravinsky game, there is no dominant strategy:
 - Strategy B is not dominant for pl. 1:
If pl. 2 chooses S, pl. 1 should choose S
 - Strategy S is also not dominant for pl. 1:
If pl. 2 chooses B, pl. 1 should choose B
- In all the examples we have seen so far, only prisoner's dilemma possesses dominant strategies

	B	S
B	(2, 1)	(0, 0)
S	(0, 0)	(1, 2)

Back to choosing a strategy...

- Hence, the question of how to choose strategies still remains for the majority of games
- **Model of rational choice:** if a player knows or has a strong belief for the choice of the other player, then he should choose the strategy that maximizes his utility
- **Thought experiment:** Suppose that someone suggests to the 2 players a strategy profile (s, t)
- When would the players be willing to follow this profile?
 - For pl. 1 to agree, it should hold that
$$u_1(s, t) \geq u_1(s', t)$$
 for every other strategy s' of pl. 1
 - For pl. 2 to agree, it should hold that
$$u_2(s, t) \geq u_2(s, t')$$
 for every other strategy t' of pl. 2

Nash Equilibria



- Definition (Nash 1950): A strategy profile (s, t) is a (pure) **Nash equilibrium**, if no player has a unilateral incentive to deviate, given the other player's choice
- This means that the following conditions should be satisfied:
 1. $u_1(s, t) \geq u_1(s', t)$ for every strategy $s' \in S^1$
 2. $u_2(s, t) \geq u_2(s, t')$ for every strategy $t' \in S^2$
- One of the dominant concepts in game theory from 1950s till now
- Most other concepts in noncooperative game theory are variations/extensions/generalizations of Nash equilibria

Pictorially:

t

	(\quad , \quad)	(\quad , \quad)	(x_1, \quad)	(\quad , \quad)	(\quad , \quad)
	(\quad , \quad)	(\quad , \quad)	(x_2, \quad)	(\quad , \quad)	(\quad , \quad)
	(\quad , \quad)	(\quad , \quad)	(x_3, \quad)	(\quad , \quad)	(\quad , \quad)
s	(\quad , y_1)	(\quad , y_2)	(x, y)	(\quad , y_4)	(\quad , y_5)
	(\quad , \quad)	(\quad , \quad)	(x_5, \quad)	(\quad , \quad)	(\quad , \quad)

In order for (s, t) to be a Nash equilibrium:

- x must be greater than or equal to any x_i in column t
- y must be greater than or equal to any y_j in row s

Nash Equilibria

- We should think of Nash equilibria as “stable” profiles of a game
 - At an equilibrium, each player thinks that if the other player does not change her strategy, then he also does not want to change his own strategy
- Hence, no player would regret for his choice at an equilibrium profile (s, t)
 - If the profile (s, t) is realized, pl. 1 sees that he did the best possible, against strategy t of pl. 2,
 - Similarly, pl. 2 sees that she did the best possible against strategy s of pl. 1
- **Attention:** If both players decide to change simultaneously, then we may have profiles where they are both better off

Examples of finding Nash equilibria in simple games

Example 1: Prisoner's Dilemma

In small games, we can examine all possible profiles and check if they form an equilibrium

- (C, C): both players have an incentive to deviate to another strategy
- (C, D): pl. 1 has an incentive to deviate
- (D, C): Same for pl. 2
- (D, D): Nobody has an incentive to change

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

Hence: The profile (D, D) is the unique Nash equilibrium of this game

– Recall that D is a dominant strategy for both players in this game

Corollary: If s is a dominant strategy of pl. 1, and t is a dominant strategy for pl. 2, then the profile (s, t) is a Nash equilibrium

Example 2: Bach or Stravinsky (BoS)

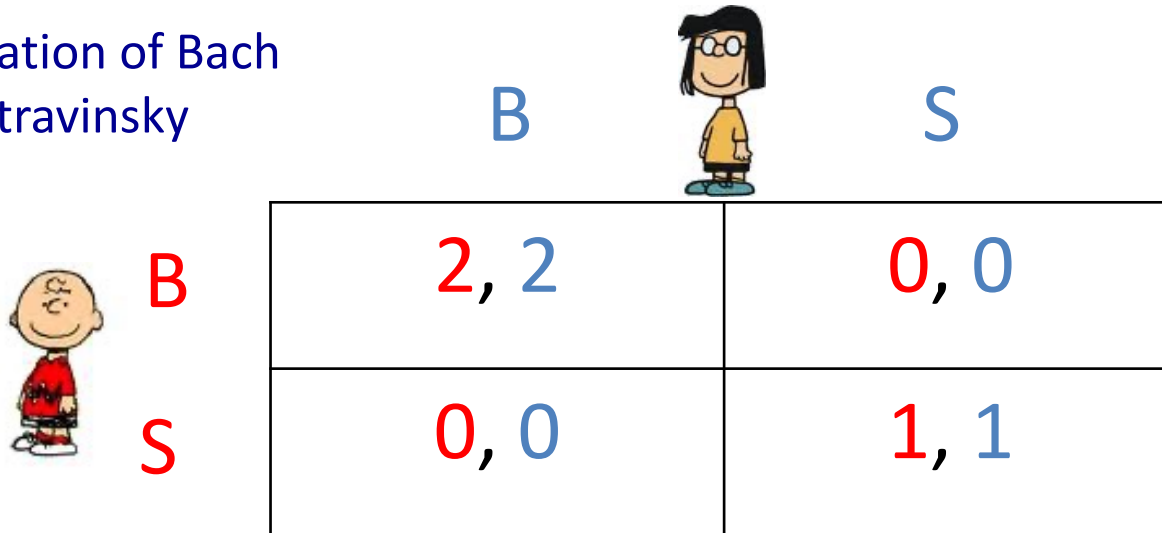
		Player 2	
		B	S
Player 1	B	2, 1	0, 0
	S	0, 0	1, 2

2 Nash equilibria:

- (B, B) and (S, S)
- Both derive the same total utility (3 units)
- But each player has a preference for a different equilibrium

Example 2a: Coordination games

Variation of Bach
or Stravinsky



	B	S
B	2, 2	0, 0
S	0, 0	1, 1

Again 2 Nash equilibria:

- (B, B) and (S, S)
- But now (B, B) is clearly the most preferable for both players
- Still the profile (S, S) is a valid equilibrium, no player has a unilateral incentive to deviate
 - At the profile (S, S), both players could deviate together in order to reach a better outcome

Example 3: The Hawk-Dove game

		
	2, 2	0, 4
	4, 0	-1, -1

- The most fair solution (D, D) is not an equilibrium
- 2 Nash equilibria: (D, H), (H, D)
- We have a stable situation only when one population dominates or destroys the other

Example 4: Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

- In every profile, some player has an incentive to deviate
- There is no Nash equilibrium!
- Note: The same is true for Rock-Paper-Scissors

Mixed strategies in games

Existence of Nash equilibria

- We saw that not all games possess Nash equilibria
- E.g. Matching Pennies, Rock-Paper-Scissors, and many others
- What would constitute a good solution in such games?

Example of a game without equilibria: Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

- In every profile, some player has an incentive to change
- Hence, no Nash equilibrium!

Q: How would we play this game in practice?

A: Maybe randomly

Matching Pennies: Randomized strategies

	$\frac{1}{2}$	$\frac{1}{2}$
	H	T
$\frac{1}{2}$ H	1, -1	-1, 1
$\frac{1}{2}$ T	-1, 1	1, -1

- Main idea: Enlarge the strategy space so that players are allowed to play non-deterministically
- Suppose both players play
 - H with probability $\frac{1}{2}$
 - T with probability $\frac{1}{2}$
- Then every outcome has a probability of $\frac{1}{4}$
- For pl. 1:
 - $P[\text{win}] = P[\text{lose}] = \frac{1}{2}$
 - Average utility = 0
- Similarly for pl. 2

Mixed strategies

- Definition: A **mixed strategy** of a player is a probability distribution on the set of his available choices
- If $S = (s_1, s_2, \dots, s_n)$ is the set of available strategies of a player, then a mixed strategy is a vector in the form
 $\mathbf{p} = (p_1, \dots, p_n)$, where
 $p_i \geq 0$ for $i=1, \dots, n$, and $p_1 + \dots + p_n = 1$
- p_j = probability for selecting the j -th strategy
- We can write it also as $p_j = p(s_j)$ = prob/ty of selecting s_j
- Matching Pennies: the uniform distribution can be written as
 $\mathbf{p} = (1/2, 1/2)$ or $p(H) = p(T) = 1/2$

Pure and mixed strategies

- From now on, we refer to the available choices of a player as *pure strategies* to distinguish them from mixed strategies
- Consider a 2-player game with $S^1 = \{s_1, s_2, \dots, s_n\}$ and $S^2 = \{t_1, t_2, \dots, t_m\}$
- Every pure strategy can also be represented as a mixed strategy that gives probability 1 to only a single choice
- E.g., the pure strategy s_1 can also be written as the mixed strategy $(1, 0, 0, \dots, 0)$
- More generally: each strategy s_i can be written in vector form as the mixed strategy $e^i = (0, 0, \dots, 1, 0, \dots, 0)$
 - 1 at position i , 0 everywhere else
 - Some times, it is convenient in the analysis to use the vector form for a pure strategy

Utility under mixed strategies

- Suppose that each player has chosen a mixed strategy in a game
- How does a player now evaluate the outcome of a game?
- We will assume that each player cares for his expected utility
 - Justified when games are played repeatedly
 - Not justified for more risk-averse or risk-seeking players

Expected utility (for 2 players)

- Consider a $n \times m$ game
- Pure strategies of pl. 1: $S^1 = \{s_1, s_2, \dots, s_n\}$
- Pure strategies of pl. 2: $S^2 = \{t_1, t_2, \dots, t_m\}$
- Let $\mathbf{p} = (p_1, \dots, p_n)$ be a mixed strategy of pl. 1 and $\mathbf{q} = (q_1, \dots, q_m)$ be a mixed strategy of pl. 2
- Expected utility of pl. 1:

$$u_1(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^n \sum_{j=1}^m p_i \cdot q_j \cdot u_1(s_i, t_j) = \sum_{i=1}^n \sum_{j=1}^m p(s_i) \cdot q(t_j) \cdot u_1(s_i, t_j)$$

- Similarly for pl. 2 (replace u_1 by u_2)

Example

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

- Let $\mathbf{p} = (4/5, 1/5)$,
 $\mathbf{q} = (1/2, 1/2)$
- $u_1(\mathbf{p}, \mathbf{q}) = 4/5 \times 1/2 \times 2 + 1/5 \times 1/2 \times 1 = 0.9$
- $u_2(\mathbf{p}, \mathbf{q}) = 4/5 \times 1/2 \times 1 + 1/5 \times 1/2 \times 2 = 0.6$
- When can we have an equilibrium with mixed strategies?

Nash equilibria with mixed strategies

- Definition: A profile of mixed strategies (\mathbf{p}, \mathbf{q}) is a **Nash equilibrium** if
 - $u_1(\mathbf{p}, \mathbf{q}) \geq u_1(\mathbf{p}', \mathbf{q})$ for any other mixed strategy \mathbf{p}' of pl. 1
 - $u_2(\mathbf{p}, \mathbf{q}) \geq u_2(\mathbf{p}, \mathbf{q}')$ for any other mixed strategy \mathbf{q}' of pl. 2
- Again, we just demand that no player has a unilateral incentive to deviate to another strategy
- How do we verify that a profile is a Nash equilibrium?
 - There is an infinite number of mixed strategies!
 - Infeasible to check all these deviations

Nash equilibria with mixed strategies

- **Corollary:** It suffices to check only deviations to pure strategies
 - Because each mixed strategy is a convex combination of pure strategies
- Equivalent definition: A profile of mixed strategies (\mathbf{p}, \mathbf{q}) is a **Nash equilibrium** if
 - $u_1(\mathbf{p}, \mathbf{q}) \geq u_1(\mathbf{e}^i, \mathbf{q})$ for every pure strategy \mathbf{e}^i of pl. 1
 - $u_2(\mathbf{p}, \mathbf{q}) \geq u_2(\mathbf{p}, \mathbf{e}^j)$ for every pure strategy \mathbf{e}^j of pl. 2
- Hence, we only need to check $n+m$ inequalities as in the case of pure equilibria

Multi-player games

Games with more than 2 players

- All the definitions we have seen can be generalized for multi-player games
 - Dominant strategies, Nash equilibria
- But: we can no longer have a representation with 2-dimensional arrays
- For n -player games we would need n -dimensional arrays (unless there is a more concise representation)

Definitions for n-player games

Definition: A game in normal form consists of

- A set of players $N = \{1, 2, \dots, n\}$
- For every player i , a set of available pure strategies S^i
- For every player i , a utility function
 $u_i: S^1 \times \dots \times S^n \rightarrow \mathbb{R}$
- Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ be a profile of **mixed strategies** for the players
- Each \mathbf{p}_i is a probability distribution on S^i
- Expected utility of pl. i under $\mathbf{p} =$

$$u_i(\mathbf{p}_1, \dots, \mathbf{p}_n) = \sum_{(s_1, \dots, s_n) \in S^1 \times \dots \times S^n} \mathbf{p}_1(s_1) \dots \mathbf{p}_n(s_n) u_i(s_1, \dots, s_n)$$

Notation

- Given a vector $s = (s_1, \dots, s_n)$, we denote by s_{-i} the vector where we have removed the i -th coordinate:

$$s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$$

- E.g., if $s = (3, 5, 7, 8)$, then
 - $s_{-3} = (3, 5, 8)$
 - $s_{-1} = (5, 7, 8)$
- We can write a strategy profile s as $s = (s_i, s_{-i})$

Definitions for n-player games

- A strategy \mathbf{p}_i of pl. i is *dominant* if
$$u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \geq u_i(e^j, \mathbf{p}_{-i})$$
for every pure strategy e^j of pl. i , and every profile \mathbf{p}_{-i} of the other players

- Replace \geq with $>$ for strictly dominant

- A profile $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ of mixed strategies is a *Nash equilibrium* if for every player i and every pure strategy e^j of pl. i , we have

$$u_i(\mathbf{p}) \geq u_i(e^j, \mathbf{p}_{-i})$$

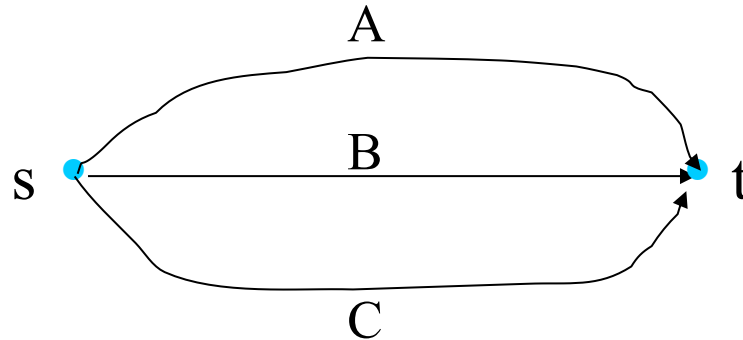
- As in 2-player games, it suffices to check only deviations to pure strategies

Nash equilibria in multi-player games

At a first glance:

- Even finding pure Nash equilibria looks already more difficult than in the 2-player case
- We can try with brute force all possible profiles
- Suppose we have n players, and each of them has m strategies: $|S^i| = m$
- There are m^n pure strategy profiles!
- However, in some cases, we can exploit symmetry or other properties to reduce our search space

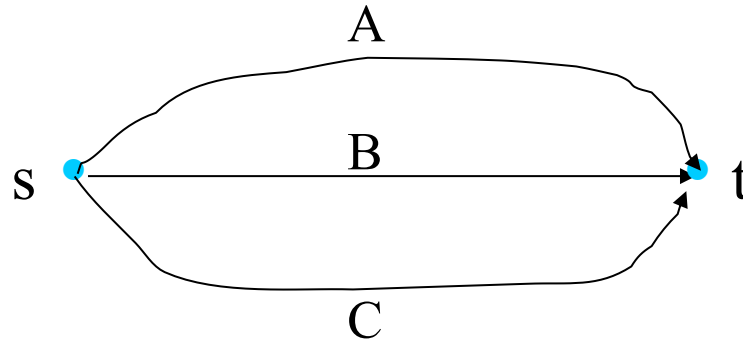
Example: Congestion games



A simple example of a congestion game:

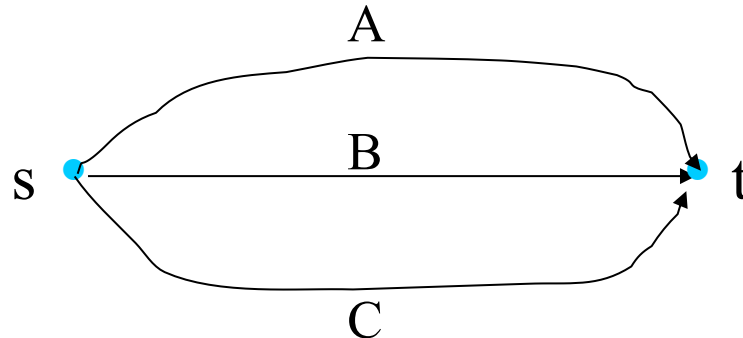
- A set of network users wants to move from s to t
- 3 possible routes, A, B, C
- Time delay in a route: depends on the number of users who have chosen this route
- $d_A(x) = 5x$, $d_B(x) = 7.5x$, $d_C(x) = 10x$,

Example: Congestion games



- Suppose we have $n = 5$ players
- For each player i , $S^i = \{A, B, C\}$
- Number of possible pure strategy profiles: $3^5 = 243$
- **Utility function of a player:** should increase when delay decreases (e.g., we can define it as $u = -\text{delay}$)
- At profile $s = (A, C, A, B, A)$
 - $u_1(s) = -15, u_2(s) = -10, u_3(s) = -15, u_4(s) = -7.5, u_5(s) = -15$

Example: Congestion games



- There is no need to examine all 243 possible profiles to find a pure equilibrium
- Exploiting symmetry:
 - In every route, the delay does not depend on who chose the route but only how many did so
- We can also exploit further properties
 - E.g. There can be no equilibrium where one of the routes is not used by some player

Homework: Find the pure Nash equilibria of this game (if there are any)

Game simplifications: Strict and weak domination

Strictly dominated strategies

- In Prisoner's dilemma, we saw that strategy D is dominant
- Strategy C is “dominated” by D
- Definition: A (pure or mixed) strategy p_i of pl. i **strictly dominates** some other strategy p' if for every profile p_{-i} of the other players, it holds that

$$u_i(p_i, p_{-i}) > u_i(p', p_{-i})$$

- Strategy p' will be called strictly dominated
- **Observation**: it suffices to consider only profiles p_{-i} with pure strategies

Strictly dominated strategies

- Strictly dominated strategies cannot be used in any Nash equilibrium
- Hence, we can remove them and reduce the size of the game
- In some cases, this results in much simpler games to analyze

Iterated Elimination of Strictly Dominated Actions

- Action **B** of **player 1** is dominated by **T** or **C**
- None of the actions of **player 2** is dominated
- If **player 1** is rational, she would never play **B**

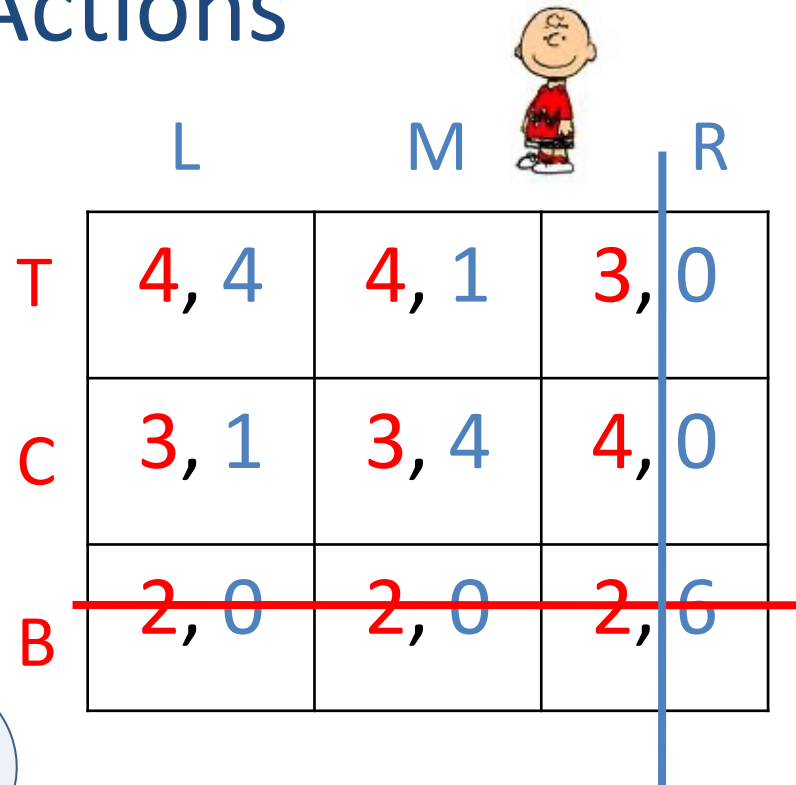
	L	M	R
T	4, 4	4, 1	3, 0
C	3, 1	3, 4	4, 0
B	2, 0	2, 0	2, 6

I should not play **B**



Iterated Elimination of Strictly Dominated Actions

- If **player 2** knows **player 1** is rational, he can assume **player 1** does not play **B**
 - then **player 2** should not play **R**

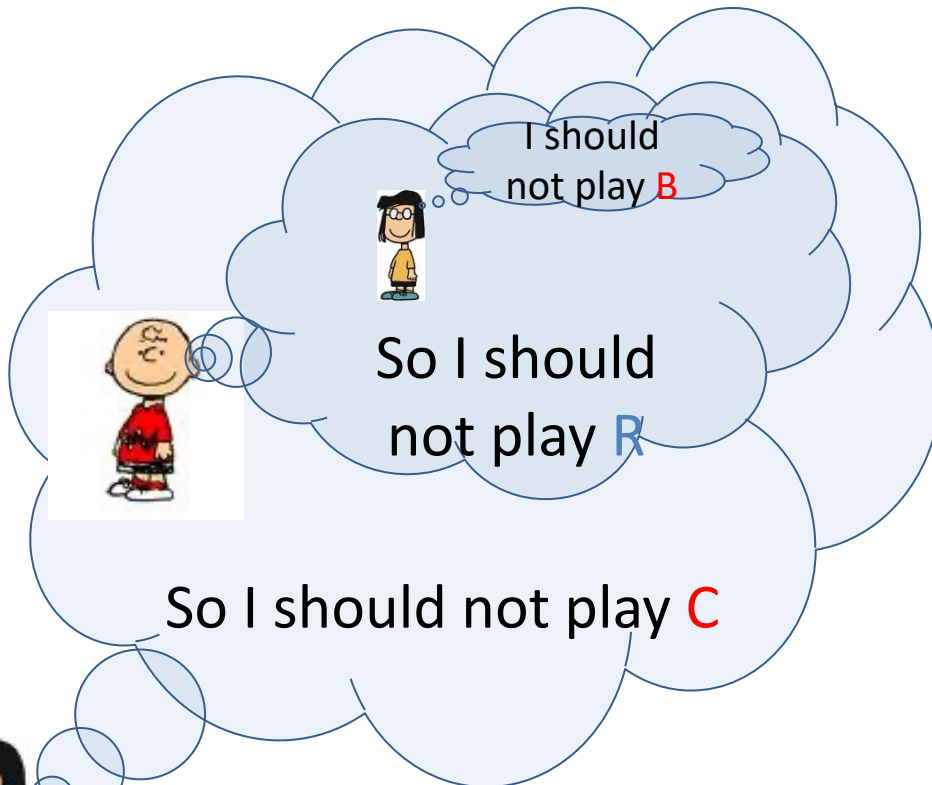


	L	M	R
T	4, 4	4, 1	3, 0
C	3, 1	3, 4	4, 0
B	2, 0	2, 0	2, 0

I should not play **B**

So I should not play **R**

Iterated Elimination of Strictly Dominated Actions



	L	M	R
T	4, 4	4, 1	3, 0
C	3, 1	3, 4	4, 0
B	2, 0	2, 0	2, 6

Unique Nash equilibrium: (T, L)

Strict domination by mixed strategies

- **Attention:** It is possible that some strategy is not strictly dominated by a pure strategy but it is dominated by a mixed strategy
- Strategy **B** of pl. 1 is not strictly dominated neither by **T** nor by **C**
- But, it is strictly dominated by the mixed strategy $(1/2, 1/2, 0)$, i.e., $0.5T + 0.5C$:
 - Proof: Consider some arbitrary strategy of pl. 2 $\mathbf{q} = (q_1, 1-q_1)$
 - $u_1(\mathbf{B}, \mathbf{q}) = 2$
 - $u_1((1/2, 1/2, 0), \mathbf{q}) = 1/2 \times q_1 \times 5 + 1/2 \times (1-q_1) \times 5 = 2.5 > 2$

	L	R
T	5, 5	0, 0
C	0, 0	5, 5
B	2, 0	2, 0

Iterated Elimination of Strictly Dominated Actions

- Given: an n -player game
 - pick a player i that has a **strictly dominated** pure strategy (dominated either by a pure or mixed strategy)
 - Remove **one** of the strictly dominated strategies of pl. i
 - repeat until **no player** has a strictly dominated pure strategy
- Facts:
 - the set of surviving actions is **independent** of the elimination order, i.e., which player was picked at each step
 - Iterated elimination of strictly dominated actions cannot destroy Nash equilibria

Weakly dominated strategies

- Definition: A (pure or mixed) strategy \mathbf{p}_i of pl. i **weakly dominates** some other strategy \mathbf{p}' if for every profile \mathbf{p}_{-i} of the other players, it holds that

$$u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \geq u_i(\mathbf{p}', \mathbf{p}_{-i})$$

and for at least one profile \mathbf{p}_{-i} we have

$$u_i(\mathbf{p}_i, \mathbf{p}_{-i}) > u_i(\mathbf{p}', \mathbf{p}_{-i})$$

- Strategy \mathbf{p}' will be called weakly dominated

Weakly dominated strategies

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

	L	R
T	2, 2	3, 0
B	0, 3	3, 3

- When we remove weakly dominated strategies, we may lose some Nash equilibria
- In the above games:
 - Strategy T weakly dominates B
 - Strategy L weakly dominates R
 - but (B, R) is an equilibrium
- Observation: In the 2nd game, we even have a better value for both players when they choose weakly dominated strategies

Iterated Elimination of Weakly Dominated Actions and Nash Equilibria

- The elimination order matters in iterated deletion of weakly dominated strategies
- Each order may eliminate a different subset of Nash equilibria
- Can we lose all equilibria of the original game?
- Theorem: For every game where each player has a finite strategy space, there is always at least one equilibrium that survives iterated elimination of weakly dominated strategies
 - thus: if we care for finding just one Nash equilibrium, no need to worry about elimination order

Exercise

	t_1	t_2
s_1	3, 2	2, 2
s_2	1, 1	0, 0
s_3	0, 0	1, 1

Execute all the possible ways of doing iterated elimination of weakly dominated strategies. Do we lose equilibria with this process?

Existence of Nash equilibria

Nash equilibria: Recap

Recall the problematic issues we have identified for pure Nash equilibria:

1. Non-existence: there exist games that do not possess an equilibrium with pure strategies
2. Non-uniqueness: there are games that have many Nash equilibria
3. Welfare guarantees: The equilibria of a game do not necessarily have the same utility for the players

Have we made any progress by considering equilibria with mixed strategies?

Existence of Nash equilibria

- Theorem [Nash 1951]: Every finite game possesses at least one equilibrium when we allow mixed strategies
 - Finite game: finite number of players, and finite number of pure strategies per player
- **Corollary**: if a game does not possess an equilibrium with pure strategies, then it definitely has one with mixed strategies
- One of the most important results in game theory
- Nash's theorem resolves the issue of non-existence
 - By allowing a richer strategy space, existence is guaranteed, no matter how big or complex the game might be

Examples

- In Prisoner's dilemma or Bach-or-Stravinsky, there exist equilibria with pure strategies
 - For such games, Nash's theorem does not add any more information. However, in addition to pure equilibria, we may also have some mixed equilibria
- Matching-Pennies: For this game, Nash's theorem guarantees that there exists an equilibrium with mixed strategies
 - In fact, it is the profile we saw: $((1/2, 1/2), (1/2, 1/2))$
- Rock-Paper-Scissors?
 - Again the uniform distribution: $((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$

Nash equilibria: Computation

- Nash's theorem only guarantees the existence of Nash equilibria
 - Proof reduces to using Brouwer's fixed point theorem
- **Brouwer's theorem:** Let $f:D \rightarrow D$, be a continuous function, and suppose D is convex and compact. Then there exists x such that $f(x) = x$
 - Many other versions of fixed point theorems also available
- Can we design polynomial time algorithms for 2-player games?
 - After all, it seems to be only a special case of the general problem of finding fixed points