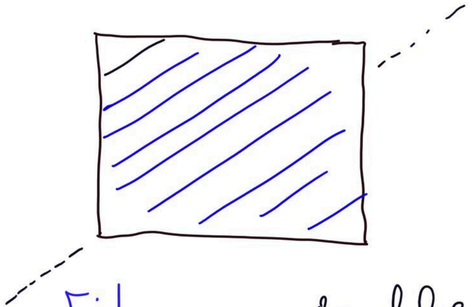
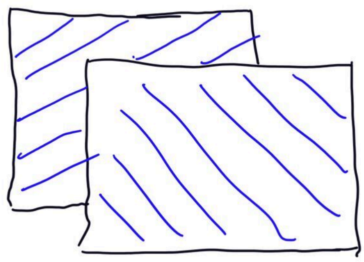
Directional models of tissues

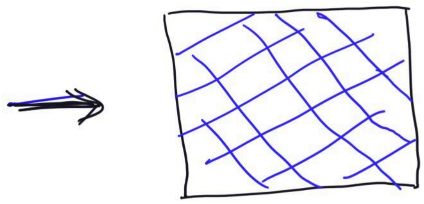
In cornea, in the eye, we have several layers of parallel fiber, where the fiber direction is different in each layer.



Fibers parallel to a symmetry axis, in a single layer.

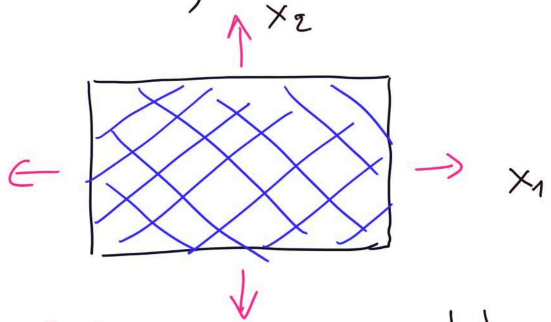


Two layers of parallel fibers, along different directions



It's like having a single layer with 2 intersecting families of fiber

That way the reinforced material (like cornea) may have substantial stiffness along different directions (~~and~~ not only along the single direction of a single family of fibers).



We end up with an orthotropic material along two perpendicular directions x_1 and x_2. The two families of fibers need not be perpendicular to each other. Obviously along other rotated directions of loading, the material will have different material properties, A material with two families of parallel, but not perpendicular to each other fibers, is called clinotropic.

Continuum planar approach of a material consisting of a matrix reinforced with fibers

For the inner main layer of the cornea, called stroma, we can write for the strain energy density function

$$\underline{W}_{stroma} = \underline{W}_{matrix}(\vec{x}) + \int_0^{\pi} \underline{W}_{fibrile}(\vec{x}, \underline{\theta}) D(\vec{x}, \underline{\theta})\, d\underline{\theta} \quad (1)$$

$\underline{W}_{matrix}$, $\underline{W}_{fibril}$: contributions (additive) of matrix and a single fiber, to the whole $\underline{W}_{stroma}$.

$\vec{x}$: position in the medium

$\underline{\theta}$: direction of a single fiber

$D(\vec{x}, \underline{\theta})$: probability function associated with the possibility of a fiber being at $\underline{\theta}$ direction at a point $\vec{x}$ in the entire medium.

A chain of rod-like cells

N_c: number of cells in the chain

V_0: volume of the medium where the cells are embedded in (like ECM containin fibroblasts)

l_0: length of the rod-like cell

F: force each cell sustrains

$\underset{\sim}{\sigma}^c$: stress tensor associated to the chain of cells

We can write the stress-strain relation

$$\underset{\sim}{\sigma}^c = \frac{N_c}{V_0} l_0 \int_{S^2} \vec{m} \otimes \vec{m} \, F(\vec{m} \cdot \underset{\sim}{\epsilon} \cdot \vec{m}) \, D(\vec{m}) \, ds \qquad (3)$$

$\vec{m} \cdot \underset{\sim}{\epsilon} \cdot \vec{m} = \frac{l}{l_0} - 1$ is the strain along the cell rod direction

l: the deformed length of a cell

We may write ~~of~~ a viscoelastic constitutive equation, like

$$\frac{d\underset{\sim}{\sigma}^c}{dt} + \frac{\underset{\sim}{\sigma}^c}{\tau_c} = \frac{3\sigma_0}{\tau_c} \underset{\sim}{A} + \left(\kappa \frac{d}{dt} + \frac{w}{\tau_c}\right) \underset{\sim}{B} : \underset{\sim}{\epsilon} \qquad (4)$$

where $\sigma_0, \tau_c, \kappa, w$ are material and geometry dependent constants, and $\underset{\sim}{A}, \underset{\sim}{B}$ are 2^{nd} order and fourth order fabric tensors respectively. Fabric tensors are related to the microstructure of the material

The elastic response of the whole medium, is given via the relation

$$\underset{\approx}{\tau}^e = 2 \frac{\partial W}{\partial \underset{\sim}{C}} \qquad (5)$$

e: elastic

$\underset{\approx}{\tau}$: 2^{nd} order Pola-Kirchhoff stress tensor

$\underset{\sim}{C}$: Richt Cauchy-Green deformation tensor

Considering additive contributions to the strain energy, we have

$$\underline{W} = \underline{W}_{matrix} + \underline{W}_{fibril} \quad (6)$$

An isotropic $\underline{W}_{matrix}$, may look like

$$\underline{W}_{matrix} = \frac{\mu}{2}\left(\mathrm{tr}\,\underset{\sim}{C} - 3\right) + \frac{\mu}{2\gamma}\left(\left(\det \underset{\sim}{C}\right)^{-\gamma} - 1\right) \quad (7)$$

which is a neo-Hookean type isotropic strain energy density. μ and γ are material parameters found from experiments.

A transeverely anisotropic $\underline{W}_{fibril}$ may look like

$$\underline{W}_{fibril} = \alpha\left[e^{(\beta(I_4 - 1))} - \beta I_4\right] \quad (8)$$

where

α, β are material parameters obtained from experiments, and

$$I_4 = \underset{\sim}{C} : \underset{\sim}{M}_c \quad (9)$$

is a pseudo-invariant related to the strain along the fiber direction.

Fabric tensors $\underset{\sim}{G}$

~~They~~ Are used to relate the microstructure of the material, to the macroscopic mechanical response. For example, we may write

$$D(\vec{n}) = D_0 + \vec{m} \cdot \underset{\sim}{G} \cdot \vec{n} \quad (10)$$

D_0 : average value of $D(\vec{n})$. It is required to be a traceless tensor (deviator).

In the plane, we may write

$$D(\vec{m}) = \frac{1}{2\pi}\left[1 + \hat{G}\cos\left(2(\theta - \hat{\theta})\right)\right] \quad (11)$$

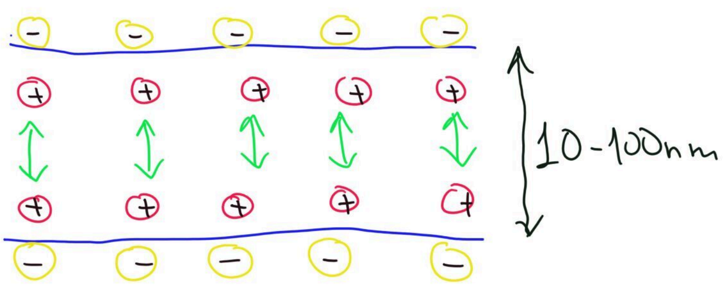
where

$\hat{G}$ are the eigenvalues of $\underset{\sim}{G}$

$\hat{\theta}$ are the associated eigenvectors of $\underset{\sim}{G}$

Elements of electr-chemo-mechanical couplings in soft tissues

Consider the ground substance in cartilage. It consists of PG's (proteoglycans) ions and fluid (water). The PG's are negatively charged proteins and are essentially immobile (the are not transported by the flowing fluid). Ions are positively ~~or~~ (Na^+, Ca^+) charged or negatively charged (Cl^-).



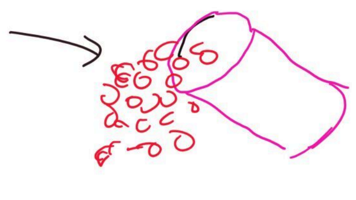
Proteoglycans (−)

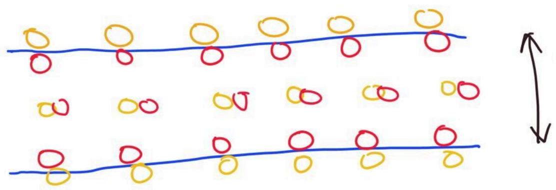
Cations (Na^+, Ca^+) (+)

−, + : negatively - positively charged

↕ : repulsive forces between the negatively charged proteoglycans

Pouring of ions





Smaller height than the initial 10-100nm

Increasing the concentration of ions in the solution, will lead to the decrease of the repulsive forces between the PG's (free swell).
The height of the cartilage strip will decrease, without applying any force to the cartilage. We have an electrochemically generated strain field (initial extension when the concentration of ions is small and subsequent contraction when the concentration of ions increases).