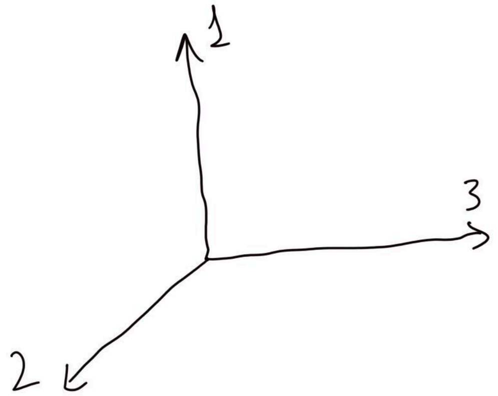
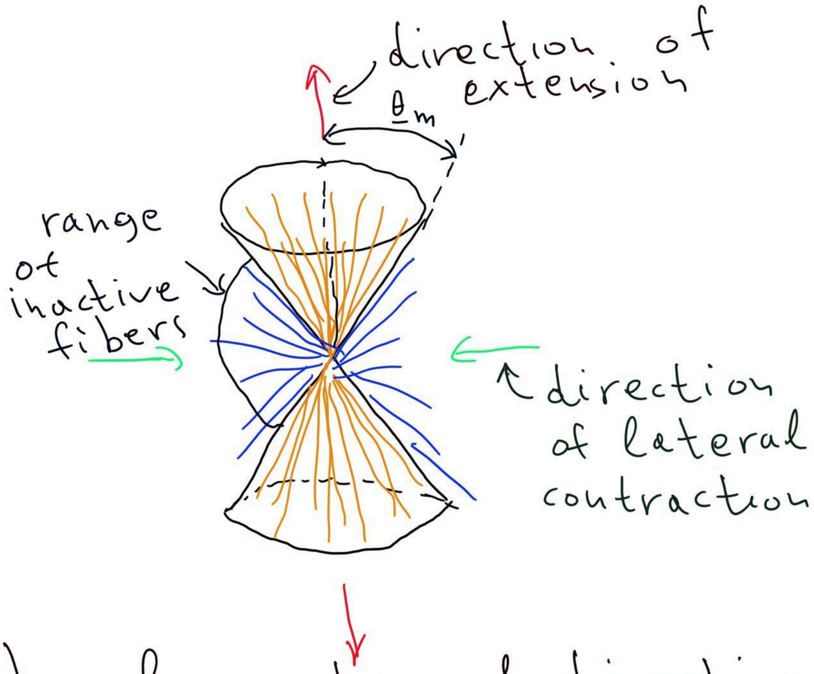
Extension of a uniformely distributed bundle of fibers, along a principal (stress and strain) direction





Loading is applied along the 1 direction.
We have uniaxial loading (stretching)
We consequently have lateral contraction alon the 2 and 3 directions.

For $\underline{\theta} \in [0, \underline{\theta}_m]$ and $\underline{\theta} \in [\pi - \underline{\theta}_m, \pi]$ ($\underline{\theta}$ is the orientation angle of each fiber), the fibers are under extension (they are activated). Fibers oriented at other angles $\underline{\theta}$, are under contraction (are inactive).

For the fibers that are in extension, we can write

$\underset{\sim}{\epsilon} : \underset{\approx}{M}_c \rightarrow$ deformation, axially, along the fiber direction $\vec{m}_c$ (remember $\underset{\sim}{M}_c = \vec{m}_c \otimes \vec{m}_c$)
we look for the directions, for which

$$\underset{\sim}{\epsilon} : \underset{\sim}{M}_c \geqslant 0 \Rightarrow \cot^2 \underline{\theta} = \frac{m_{c1}^2}{m_{c2}^2 + m_{c3}^2} \geqslant \cot^2 \underline{\theta}_m \equiv \frac{-\varepsilon_{22}}{\varepsilon_{11}}$$

from which we can find the critical angle $\underline{\theta}_m$

We have effectively created an anisotropic material, by loading the uniform bunde of fibers along a single (the 1) direction. On further loading the material will behave in an anisotropic manner, although its initial structure (uniform distributon of fibers along all ~~di~~ radial directions in a unit spher) indicated that the material is isotropic.

We write the stress-strain ~~bound~~ relations for this new material as

$$\underset{\sim}{\sigma}^c = \underset{\approx}{E} : \underset{\sim}{\epsilon} \Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} E_{11}^c & E_{12}^c & E_{12}^c \\ E_{12}^c & E_{22}^c & E_{23}^c \\ E_{12}^c & E_{23}^c & E_{22}^c \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix}$$

Upper index c indicates properties of the bundle of collagen fibers.

In the above constitutive equation for

$$E_{11}^c = \frac{K^c}{5}\left[1 - (\cos \underline{\theta}_m)^5\right]$$

which is deformation dependent since it depends on the critical activation ange $\underline{\theta}_m$. Also $K^c = n^c K_c$.

We can estimate the efficiency of a reinforcement from the ratio of ~~the~~ a Young's modulus of the anisotropic material over the Young's modulus of the material wint the associated isotropic reinforcement.

Apparent material properties of the dry tissue (e.g cartilage)

The stiffness of the cartilage will be the sum of the stiffnesses of the ground substance and the collagen fiber. That is

$$\underset{\approx}{E}^{car} = \underset{\approx}{E}^{gs} + \underset{\approx}{E}^{c}$$

The apparent moduli of the ground substance are

$$\lambda = n^{gs} \Lambda_{gs} \quad , \quad \mu = n^{gs} M_{gs}$$

n^{gs}: volume fraction of the ground substance in the cartilage

Λ_{gs}, M_{gs}: intrinsic Lamè moduli of the matrix (ground substance).

For the collagen network, we have the components of the stiffness tensor $\underset{\approx}{E}^{car}$, from the matrix constitutive equation, reffering to uniaxial stretching.

$$\underset{\sim}{\sigma} = \underset{\approx}{E}^{car} : \underset{\sim}{\epsilon} \Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} \overbrace{\lambda + 2\mu}^{\text{g.s. contribution}} + \overbrace{E^{c}_{11}}^{\text{c. contribution}} & \lambda + E^{c}_{12} & \lambda + E^{c}_{12} \\ \lambda + E^{c}_{12} & \lambda + 2\mu + E^{c}_{22} & \lambda + E^{c}_{23} \\ \lambda + E^{c}_{23} & \lambda + E^{c}_{12} & \lambda + 2\mu + E^{c}_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix}$$