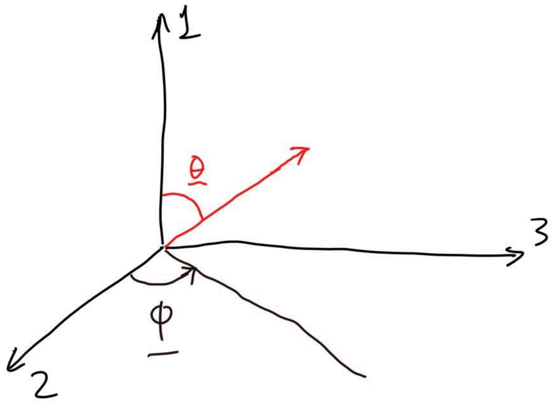
Random directional distribution of collagen fibers



The unit directional vector which indicates the direction at a point along a collagen fiber, is denoted by $\vec{m}_c$.

In the spherical coordinate system, $\vec{m}_c$ is written as:

$$\vec{m}_c = \begin{bmatrix} \cos\underline{\theta} \\ \sin\underline{\theta}\cos\underline{\phi} \\ \sin\underline{\theta}\sin\underline{\phi} \end{bmatrix}$$

where $\underline{\theta}, \underline{\phi}$ are angles between the tangent to the fiber line, in the undeformed configuration. $\vec{m}_c$ is the unit vector tangent to the point along a collagen fiber. Obviously if the fiber is straight, $\vec{m}_c$ is the unit vector along that direction. The range of the angles is $\underline{\theta} \in [0, \pi]$, $\underline{\phi} \in [0, 2\pi]$. We use also the $2^{\underline{nd}}$ order directional tensor $\underset{\sim}{M}_c = \vec{m}_c \otimes \vec{m}_c$.

The 4^{th} order tensor $\underset{\sim}{M}_c \otimes \underset{\sim}{M}_c$ may be used in order the write down the stiffness of a single fiber or of a bundle of fibers.

If we calculate the directional integration over space directions, of the $2^{\underline{nd}}$ order tensor $\underset{\sim}{M}_c$, we get

$$\int_{S^2} \vec{m} \otimes \vec{m}_c \,\frac{dS}{4\pi} = \frac{1}{4\pi}\int_0^{2\pi} d\underline{\phi} \int_0^{\pi} \vec{m}_c \otimes \vec{m}_c \sin\underline{\theta}\, d\underline{\theta} = \frac{\underset{\sim}{I}}{3} \quad (2)$$

which indicates the isotropic (ever over all radial directions within the unit sphere) distribution of the fibers.

Similarly we can perform integrations for the 4^{th} order tensor $\underset{\sim}{M}_c \otimes \underset{\sim}{M}_c$.

These integrations give us information about the overall directional properties of the whole material that covers the space. The effect of the fibers themselves is taken into account (not any matrix material where the fibers are embedded in it is considered at the moment). No interactions between the fibers is considered (like cross links between them).

Conewise linear stress-strain response of collagen fibers

The strain energy density function, for a single fiber, can be written as:

$$w_c = \frac{K_c}{2} \langle \underset{\sim}{\varepsilon} : \underset{\sim}{M_c} \rangle^2 \quad (3)$$

$$w = \frac{1}{2} E \varepsilon^2$$

$$\sigma = \frac{\partial w}{\partial \varepsilon} = E\varepsilon$$

$$\frac{\partial^2 w}{\partial \varepsilon^2} = E$$

w_c: strain energy density function for a single fiber

K_c: material constant found from experiments

$\underset{\sim}{\varepsilon}$: small strain tensor (2nd order tensor)

$\langle\ \rangle$: indicates the positive part of the quantity included in the brackets. The outcome of the $\langle\ \rangle$ is either positive or zero (cannot be negative).

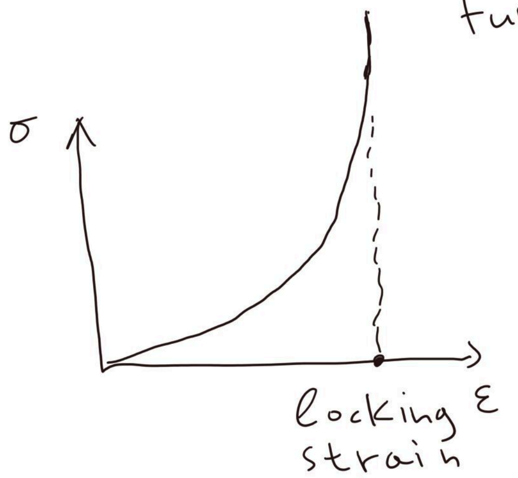
Differentiating (3) w.r.t $\underset{\sim}{\varepsilon}$, we get the stress tensor related to the strain tensor $\underset{\sim}{\varepsilon}$:

$$\frac{\partial w_c}{\partial \underset{\sim}{\varepsilon}} = K_c \langle \underset{\sim}{\varepsilon} : \underset{\sim}{M_c} \rangle \underset{\sim}{M_c} \quad (4)$$

Gives the stress associated the the strain tensor $\underset{\sim}{\varepsilon}$.

Another differentiation gives the stiffness in tensorial form

$$\frac{\partial^2 w_c}{\partial \underset{\sim}{\varepsilon}\, \partial \underset{\sim}{\varepsilon}} = K_c \underbrace{H\left(\underset{\sim}{\varepsilon} : \underset{\sim}{M_c}\right)}_{\text{discontinuity function}} \underset{\sim}{M_c} \otimes \underset{\sim}{M_c} \qquad (5)$$



Suppose now that all fibers, around all radial directions of the unit sphere, are subjected to the same extension (extension of a single magnitude). Integration over space and directions (here actually only over direction since the space part is seperated from the directional integral), gives:

For the strain energy w_c:

$$n^c \int_{S^2} w_c \frac{dS}{4\pi} = \frac{K_c}{15} \frac{1}{2} \left(\underset{\sim}{I} : \underset{\sim}{\varepsilon}\right)^2 + \underset{\sim}{\varepsilon} : \underset{\sim}{\varepsilon} \qquad (6)$$

n^c: volume fraction of the whole tissue (e.g. cartilage), occupied by the collagen fibers.

We end up with an isotropic strain energy density function, for the material consisting of all the collagen fibers around the unit sphere.

Similarly we can get the corresponding average stress tensor, from the relation

$$n^c \int_{S^2} \frac{\partial w_c}{\partial \underset{\sim}{\varepsilon}} \frac{dS}{4\pi} = \frac{K^c}{15} (\underset{\sim}{\varepsilon} : \underset{\sim}{I}) \underset{\sim}{I} + 2 \underset{\sim}{\varepsilon} \quad (7)$$

In the above expressions, the apparent Lamè moduli are (~~for a single collagen fiber~~) for all the collagen fibers

$$\lambda^c = \mu^c = \frac{K^c}{15} \quad (8)$$

where

$$K^c = n^c K_c \quad (9)$$

Quantities with upper index c, refer to the material consisting of all collagen fibers. Constant K_c refers to a single collagen fiber.

Homework: Try to prove relation (6) of the notes (this part). It is relation (12.2.17) first. in the book.