Constitutive equations for articular cartilage: Case of free swelling

In the case of free swelling, the cartilage specimen expands, due to the repulsive forces of the PG's, with no stresses developing on it (the specimen).

The general constitutive equation due to chemomechanical coupling, is

$$\underset{\sim}{\sigma} + \tilde{p}_E \underset{\sim}{I} + (\pi_{osm} + p_{ch}) \underset{\sim}{I} = W_{ch,2}(E_{mo}) \underset{\approx}{\mathbb{E}}^{car}(\underset{\sim}{\varepsilon}) : \underset{\sim}{\varepsilon} \qquad (1)$$

which may be a non-linear relation w.r.t $\underset{\sim}{\varepsilon}$.

$\underset{\sim}{\sigma}$: Cauchy stress tensor (2nd order)

$\tilde{p}_E$: pressure of the fictitious bath

$\underset{\sim}{I}$: unit 2nd order tensor

π_{osm}: osmotic pressure

p_{ch}: pressure of chemical origin

$W_{ch,2}$: chemistry dependent scalar factor

E_{mo}: mobile species in the extra-fibrillar phase ($E - \{PG's\}$)

$\underset{\approx}{\mathbb{E}}^{car}$: 4th order stiffness tensor

$\underset{\sim}{\varepsilon}$: 2nd order strain tensor

The tissue is deformed, along all directions, identically. Since the swelling is not caused by mechanical forces (the reason it's called "free"), the term

$$\underset{\sim}{\sigma} + \tilde{P}_E \underset{\sim}{I} = \underset{\sim}{0} \quad (2)$$

Note that

$$\underset{\approx}{\mathbb{E}}^{car}(\underset{\sim}{\varepsilon}) = \underset{\approx}{\mathbb{E}}^{gs} + \underset{\approx}{\mathbb{E}}^{c}(\underset{\sim}{\varepsilon}) \quad (3)$$

Since the collagen fibers are considered to be distributed randomly, along all directions, the material response is isotropic.

The normal strain along any direction, can be obtained as

$$\underset{\sim}{\varepsilon}^{ifs}_{11} = \frac{\Pi_{osm} + P_{ch}}{W_{ch,2}} \frac{1}{3K^{car}_{ifs}} \quad (4)$$

ifs: isotropic free swelling

The bulk modulus of cartilage, with all collagen fibers activated, is

$$K^{car}_{ifs} = \lambda^{gs} + \lambda^{c} + \frac{2}{3}(\mu^{gs} + \mu^{c}) \quad (5)$$

Stress-strain relation, for cartilage, under uniaxial ~~[illegible]~~ traction, along axis 1.

The boundary conditions, along the lateral directions, are

$$\left.\begin{array}{l}\sigma_{22}+\tilde{p}_E=0\\ \sigma_{33}+\tilde{p}_E=0\end{array}\right\}\quad(1)$$

We have an active σ_{11} stress. The constitutive equations are

$$\varepsilon_{11}=\frac{\Pi_{osm}+P_{ch}}{W_{ch,2}}\,\frac{1}{3K^{car}}+\frac{\sigma_{11}+\tilde{p}_E}{W_{ch,2}}\,\frac{1}{E^{car}}\quad(2)$$

$$\varepsilon_{22}=\frac{\Pi_{osm}+P_{ch}}{W_{ch,2}}\,\frac{1}{M}-\nu^{car}\varepsilon_{11}\quad(3)$$

For the dilatation (percentage ~~unit~~ volume change) we can write

$$\operatorname{tr}\underset{\sim}{\varepsilon}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\overset{(2),(3)}{=}\frac{\Pi_{osm}+P_{ch}}{W_{ch,2}}\,\frac{2}{M}+\left(1-2\nu^{car}\right)\varepsilon_{11}\quad(4)$$

We also have the equalities

$$E^{car}=E_{11}^{car}+2E_{12}^{car}\nu^{car}\quad(5)$$

$$K^{car}=\frac{E^{car}}{3(1-2\nu^{car})}\quad(6)$$

$$M^{car}=E_{22}^{car}+E_{23}^{car}\quad(7)$$

which indicate the dependence of the material constants of the cartilage, on the strains (conewise response).

Equation (2), ~~can~~ can be written as

$$\sigma_{11}+\tilde{p}_E+\left(1-2\nu^{car}\right)\left(\Pi_{osm}+P_{ch}\right)=W_{ch,2}E^{car}\varepsilon_{11}\quad(8)$$

For the hypertonic state, under uniaxial traction, we have

$$\underset{\sim}{\sigma} + \tilde{p}_E \underset{\sim}{I} = \underset{\approx}{\mathbb{E}}^{car}(\underset{\sim}{\varepsilon}) : \underset{\sim}{\varepsilon} \quad (9)$$

since $\pi^{osm} + p_{ch} = 0$ and $W_{ch,2} = 1$.

The scalar equations of (9), are

$$\sigma_{11} + \tilde{p}_E = \mathbb{E}^{car}(\underset{\sim}{\varepsilon})\, \varepsilon_{11} \quad (10)$$

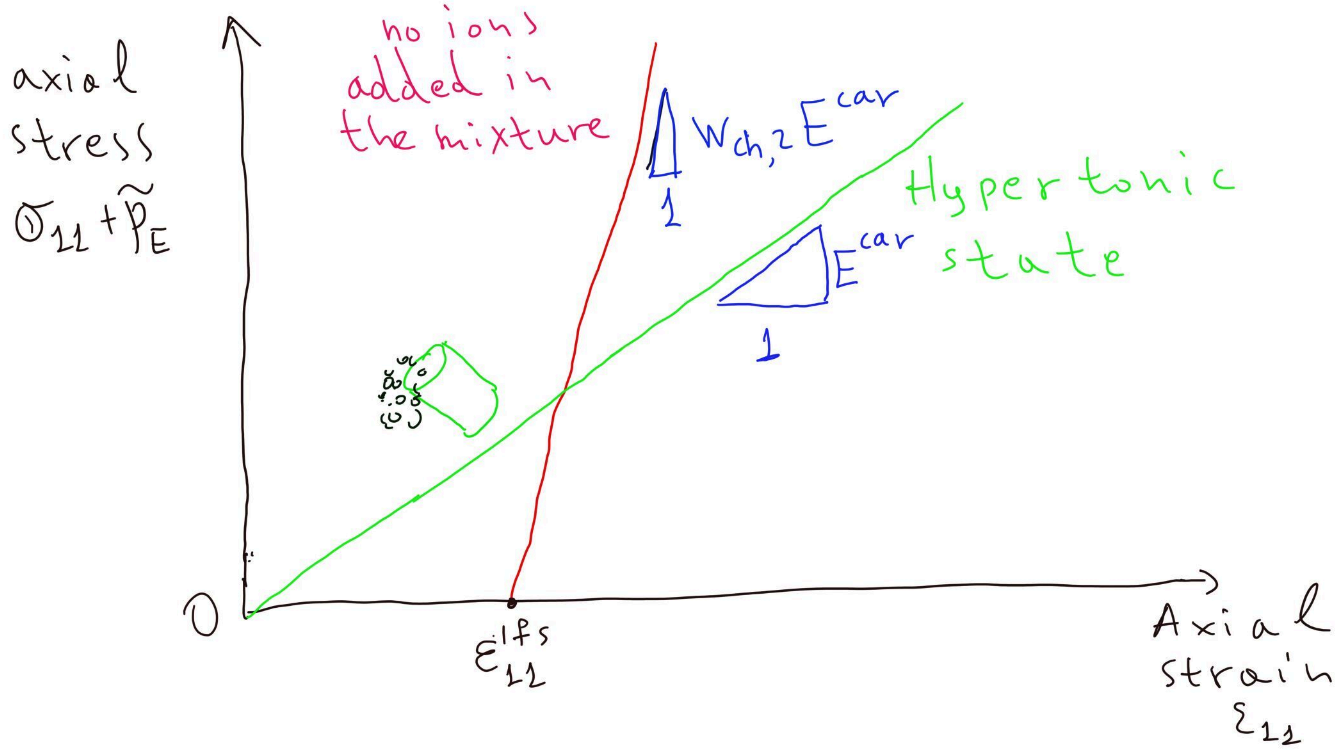
$$\sigma_{22} + \tilde{p}_E = \sigma_{33} + \tilde{p}_E = 0 \quad (11)$$

Also

$$\varepsilon_{22} = -\nu^{car}(\underset{\sim}{\varepsilon})\, \varepsilon_{11} \quad (12)$$

$$\mathrm{tr}\, \underset{\sim}{\varepsilon} = \left(1 - 2\nu^{car}(\underset{\sim}{\varepsilon})\right) \varepsilon_{11} \quad (13)$$

Stress-strain curves for uniaxial traction



We see a reduction in the slope of the stress-strain curve (more compliant cartilage), after the addition of ions in the mixture. In the hypertonic state (where we have added too many ions in the mixture), the stress-srain curve starts from the origin.

Finite element analysis of electro-chemomechanical couplings

Nature of Equation	Field Equation	Main unknowns
Balance of momentum in the whole mixture	$\operatorname{div} \underset{\sim}{\sigma} = 0$	Displacement $\vec{u}$
Balance of mass equation of the whole mixture	$\operatorname{div} \vec{V}_s + \operatorname{div} \vec{J}_{kE} = 0$	Chemical potential μ_{wE} of extrafibrillar water.
Balance of mass for EF ions	$\frac{1}{\det \underset{\sim}{F}}\left(\frac{d\nu^{kE}}{dt} + \frac{d\nu^{kI}}{dt}\right) + \operatorname{div} \vec{J}_{kE} = 0$	Electrochemical potentials μ^{ec}_{kE}, $k \in E$ ions
Balance of mass of IF-EF exchange	$\frac{1}{\det \underset{\sim}{F}} \frac{dm^{iI}}{dt} - O_i(m_{nI} - m_{nE})$	IF mass content m^{iI}, $i \in I_n$, $n \in I_{ne}$