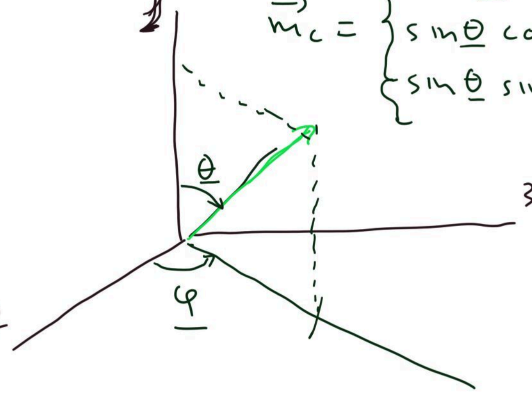
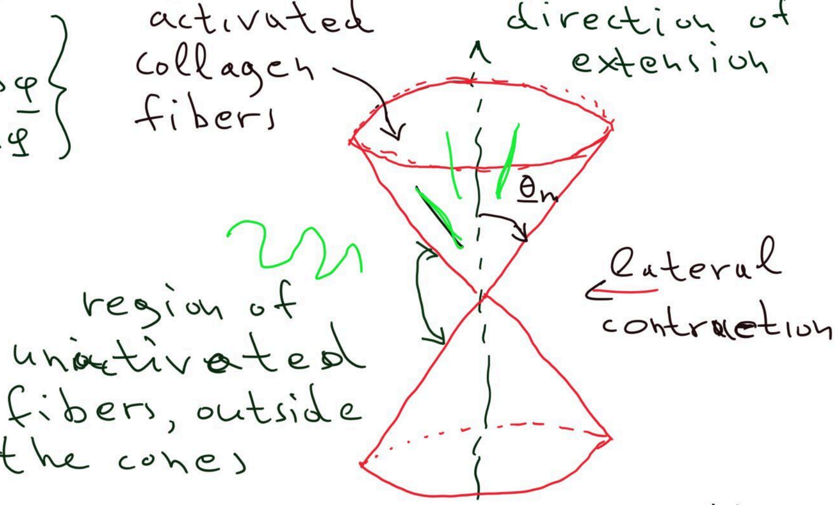
Random distribution of collagen fibers



$$\vec{m}_c = \begin{Bmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{Bmatrix}$$



θ_m: angle of the cone generator lines with the direction of extension. Fibers that lie within the cone, are activated.

For linear elastic behavior, we can have

$$w_c = \frac{K_c}{2}\langle \underset{\sim}{\epsilon} : \underset{\sim}{M}_c \rangle^2$$

$\underset{\sim}{\epsilon}$: small strain tensor

$$\frac{\partial w_c}{\partial \underset{\sim}{\epsilon}} = K_c \langle \underset{\sim}{\epsilon} : \underset{\sim}{M}_c \rangle \underset{\sim}{M}_c$$

$$\frac{\partial^2 w_c}{\partial \underset{\sim}{\epsilon}\partial \underset{\sim}{\epsilon}} = K_c \underbrace{H(\underset{\sim}{\epsilon} : \underset{\sim}{M}_c)}_{\text{discontinuous stiffness at zero strain}} \underset{\sim}{M}_c \otimes \underset{\sim}{M}_c$$

quantities for a single fiber

Suppose that all fibers are in extension.
Integrate the quantities referred to a single fiber, around the unit sphere and the take the average.

$$n^c \int_{S^2} w_c \frac{dS}{4\pi} = \frac{K^c}{15}\left[\frac{1}{2}(\underset{\sim}{I} : \underset{\sim}{\epsilon})^2 + \underset{\sim}{\epsilon} : \underset{\sim}{\epsilon}\right]$$

The integral $\times\, n^c$ gives the strain energy density, of the collagen part of the composite material, that consists of fibers embedded in a matrix.

n^c: volume density of the fibers in the composite material (fibers + matrix).

$K^c = n^c \cdot K_c$

Remember that for a uniaxial extension of a ~~bar~~, bar, cosisting of Hookean linear material

$$w = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}(E\varepsilon)\varepsilon = \frac{1}{2}E\varepsilon^2$$

Similarly, integrating around the unit sphere, for the other 2 quantities we get

$$n^c \int_{S^2} \frac{\partial w_c}{\partial \underset{\sim}{\varepsilon}} \frac{dS}{4\pi} = \frac{K^c}{15}\left[\left(\underset{\sim}{\varepsilon} : \underset{\sim}{I}\right)\underset{\sim}{I} + 2\underset{\sim}{\varepsilon}\right] \rightarrow \text{stress}$$

$$n^c \int_{S^2} \frac{\partial^2 w_c}{\partial \underset{\sim}{\varepsilon} \partial \underset{\sim}{\varepsilon}} \frac{dS}{4\pi} = \frac{K^c}{15}\left[\underset{\sim}{I} \otimes \underset{\sim}{I} + 2\,\underset{\sim}{I} \,\overline{\underline{\otimes}}\, \underset{\sim}{I}\right]$$

$\overline{\underline{\otimes}}$: gives another 4^{th} order tensor

From the above relations, we can relate the Lame constants, λ and μ, with the single fiber material constant K_c. We finally get

$$\lambda^c = \mu^c = \frac{K^c}{15}, \quad K^c = n^c K_c$$

Since $\lambda^c = \mu^c$, Poisson's ratio ~~coeff~~ takes the value 0,25.

If we have uniaxial extension along direction 1, and symmetry along direction 2 and 3, we get the activation relation

$$\underset{\sim}{\varepsilon} : \underset{\sim}{M}_c \geq 0 \Rightarrow \underbrace{\cot\underline{\theta} = \frac{m_{c_1}^2}{m_{c_2}^2 + m_{c_3}^2}}_{\text{fiber orientation relation}} \geq \underbrace{\cot\underline{\theta}_m = -\frac{\epsilon_{22}}{\epsilon_{11}}}_{\text{strain field relation}}$$

This is how $\underline{\theta}_m$ can be evaluated.