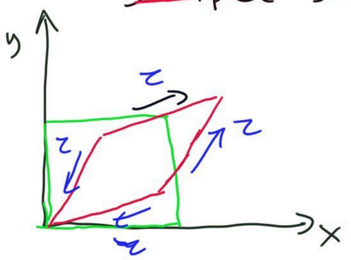
Directional averaging of fiber-reinforced tissues

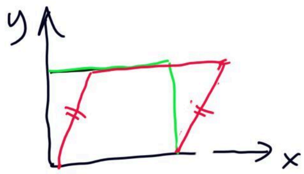
Most soft tissues consist of fiber chains (either molecular chains (like collagen) or cellular tissues (like in blood vessels)). These chains govern the anisotropy of the tissue. We can use the presence of these chains in order to describe the macroscopic mechanical behavior of the tissue. In order to achieve this we have to do some kind of homogenisation, in order to describe the mechanical response of the entire material (matrix plus fibers that are embedded in the matrix).

~~Simple~~ Pure shear: the lengths of the edges do not change. The angles between them change.



~~Direct~~ Simple shear: The // lines undergo change of length.

This Shear cannot exist without acompanying tension.



Averaging of a material property $f(\mathcal{V}, \vec{m}, \vec{X})$

f: the property $\quad$ $\mathcal{V}$: set of variables

$\vec{m}$: fiber direction at a point in space, with position vector $\vec{X}$ $\quad$ $\vec{X}$: position vector

$$\int_V \int_{S^2} f(\mathcal{V}, \vec{m}, \vec{X}) \frac{dS}{4\pi} \frac{dV(\vec{X})}{V}$$

S^2: the surface of the unit sphere (sphere with radius = 1.

V: the volume of the Representative Elementary Volume (REV) of the material.

For plane problems

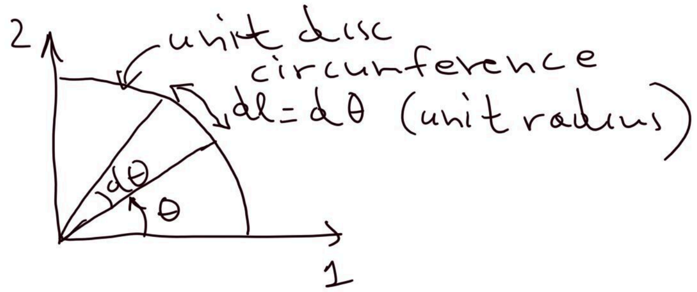
$$\vec{m} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

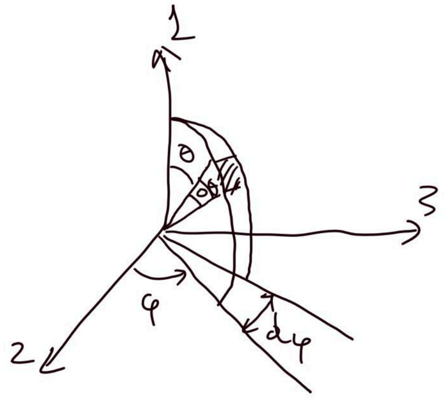
In the simplest case $f = f(\vec{m})$.

In 3d

$$\vec{m} = \begin{bmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{bmatrix}$$

In the plane





The averaging in our case is used for the modelling of soft tissues containing collagen fibers.

<u>A single family of parallel collagen fibers</u>

$\vec{m}_c$: directional unit vector of a collagen fiber.
We use the strain energy density function

$$w_c = w_c(L^2) \quad (1)$$

with

$$L = \frac{1}{2}\langle \underset{\sim}{C} : \underset{\sim}{M}_c - 1\rangle = \langle \underset{\sim}{E}_G : \underset{\sim}{M}_c\rangle \quad (2)$$

$\underset{\sim}{C}$: right Cauchy-Green deformation tensor

$\underset{\sim}{M}_c = \vec{m}_c \otimes \vec{m}_c$

$\underset{\sim}{E}_G$: Lagrangian strain tensor with

$$\underset{\sim}{E}_G = \frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I})$$

$\underset{\sim}{I}$: identity 2^{nd} order tensor

Also $\underset{\sim}{C} = \underset{\sim}{F}^T \cdot \underset{\sim}{F}$ where $\underset{\sim}{F} = \frac{d\vec{x}}{d\vec{X}}$.

The angular brackets $\langle\rangle$ indicate the positive part of the quantity that is inside the $\langle\rangle$.

We use the L^2 within the $\langle\rangle$ in order to ensure continuity of the stress-strain curve. On the other hand, continuity of the stiffness, may or may not be ensured.

Strain energy density: $w_c = \frac{K_c}{2k_c}\left[e^{k_c L^2} - 1\right]$ (1)

K_c: material constant (MPa)

k_c: dimensionless constant

Stress 2nd Piola-Kirchoff: $\frac{\partial w_c}{\partial \underset{\sim}{E}_G} = K_c e^{k_c L^2} L \underset{\sim}{M}_c$

Stress $\to \infty$ for strain $\to \infty$

Stiffness: $\frac{\partial w_c}{\partial \underset{\sim}{E}_G \partial \underset{\sim}{E}_G} = K_c e^{k_c L^2}\left(2k_c L^2 + H(x)\right) \underset{\sim}{M}_c \otimes \underset{\sim}{M}_c$

$2k_cL^2$: Continuous part

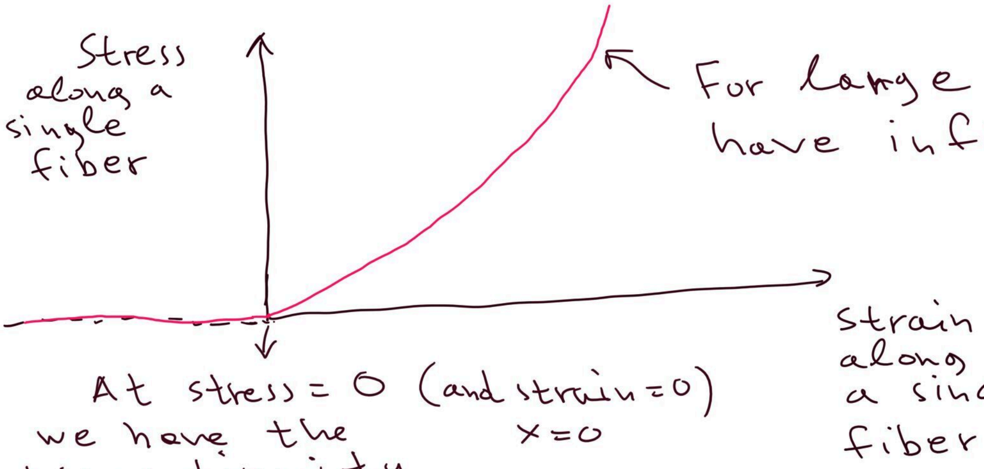
$H(x)$: Discontinuous part

$x = \underset{\sim}{M}_c : \underset{\sim}{E}_G$

$\underset{\sim}{M}_c = \vec{m}_c \otimes \vec{m}_c$

$\vec{m}_c$: unit vector along a fiber

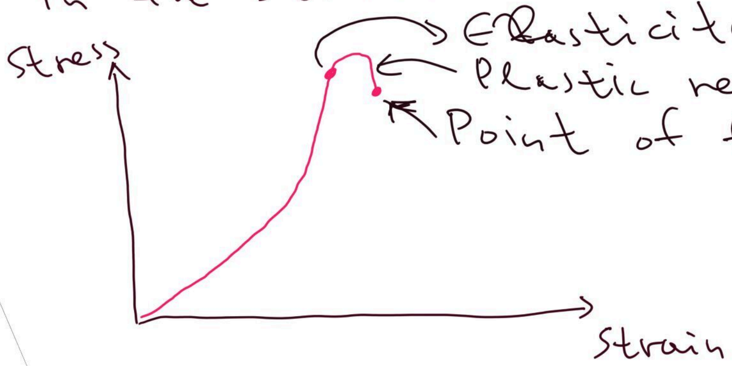
x: strain along the fiber



For large strains we have infinite stiffness.

At stress = 0 (and strain = 0) we have the discontinuity in the stiffness

$x = 0$



Elasticity limit
Plastic region
Point of fracture

For a single fiber

We can use a modified form of w_c, in order to obtain a non-infinite (actually in this case a constant stiffness), for very large strains. We choose a w_c of the form

$$w_c = (1 - H(x_2))\, w_{cf_1} + H(x_2)\, w_{cf_2}$$

Accordingly we obtain

$$\underset{\approx}{\Sigma} = (1 - H(x_2))\, \underset{\approx}{\Sigma}_1 + (1 - H(x_2))\, \underset{\approx}{\Sigma}_2$$

$\underset{\approx}{\Sigma}$: 2nd Piola-Kirchoff stress tensor

$$\underset{\approx}{E} = (1-H(x_2))\,\underset{\approx}{E}_1 + (1-H(x_2))\,\underset{\approx}{E}_2$$

$\underset{\approx}{E}$: stiffness tensor (4th order)

$$w_{cf_1} = \frac{K_c}{2k_c}\left[e^{k_c L_1^2} - 1\right]$$

$$w_{cf_2} = \underline{\underline{\tau}}_t : \underset{\sim}{E}_G + \frac{1}{2}\left[\underset{\sim}{E}_G - \underset{\sim}{E}_{Gt}\right] : \underset{\approx}{E}_t : \left[\underset{\sim}{E}_G - \underset{\sim}{E}_{Gt}\right]$$

Index t refers to a transition

$$\underline{\underline{\tau}}_1 = \frac{\partial w_{cf_1}}{\partial \underset{\sim}{E}_G} = K_c e^{k_c L^2} L_1 \underset{\sim}{M}_c$$

$$\underset{\approx}{E}_1 = \frac{\partial w_{cf_1}}{\partial \underset{\sim}{E}_G \partial \underset{\sim}{E}_G} = K_c e^{k_c L_1^2}\left(2k_c L_1^2 + H(x_1)\right) \underset{\sim}{M}_c \otimes \underset{\sim}{M}_c$$

$$\underset{\approx}{\tau}_2 = \frac{\partial w_{cf_2}}{\partial \underset{\sim}{E}_G} = \underline{\underline{\tau}}_t + \underset{\approx}{E}_t : \left(\underset{\sim}{E}_G - \underset{\sim}{E}_{Gt}\right)$$

$$\underset{\approx}{E}_2 = \frac{\partial w_{cf_2}}{\partial \underset{\sim}{E}_G \partial \underset{\sim}{E}_G} = \underset{\approx}{E}_T \longrightarrow$$ constant stiffness after the transition point

$$L_1 = \langle \underbrace{\underset{\sim}{E}_G : \underset{\sim}{M}_c}_{x_1} \rangle, \quad L_2 = \langle \underbrace{(\underset{\sim}{E}_G - \underset{\sim}{E}_{GT}) : \underset{\sim}{M}_c}_{x_2} \rangle$$

$$\underline{\underline{\tau}}_t = \underline{\underline{\tau}}_1(\underset{\sim}{E}_{Gt}), \quad \underset{\approx}{E}_t = \underset{\approx}{E}_1(\underset{\sim}{E}_{Gt})$$