For a single family of fibers, where all fibers are parallel and aligned along a single direction, we use the following strain invariants (and pseudo-invariants), of the $\underset{\sim}{C}$ strain tensor:

$$I_1 = tr\,\underset{\sim}{C}$$

$$I_2 = \frac{1}{2}\left(tr^2\underset{\sim}{C} - tr\,\underset{\sim}{C}^2\right)$$

$$I_3 = \det \underset{\sim}{C}$$

$$I_4 = \underset{\sim}{C} : \underset{\sim}{M}$$

$$I_5 = \underset{\sim}{C}^2 : \underset{\sim}{M}$$

I_4: expresses the square of the stretch along the fiber direction

I_4 is related to the extension of a fiber in the direction indicated by the structure tensor $\underset{\sim}{M}$ in the undeformed configuration

$\underset{\sim}{M} = \vec{m} \otimes \vec{m}$: $\underset{\sim}{M}$ is a second order structure tensor, related to the unit direction vector $\vec{m}$ of a fiber at a particular point in the medium

For the deformed configuration, the corresponding second order structure tensor is

$$\underset{\sim}{M} = \vec{m} \otimes \vec{m} = \frac{\underset{\sim}{F} \cdot \underset{\sim}{M} \cdot \underset{\sim}{F}^{-T}}{|\underset{\sim}{F} \cdot \vec{m}|^2}$$

$\vec{m}$: unit directional vector of the fiber in the deformed configuration

$\underset{\sim}{F}$: deformation gradient tensor.

I_5 pseudo-invariant is related either to shear between the fiber and the matrix and is also related to possible interaction (link) between parallel fibers.

The deformation of the fibers is the same as the deformation of the matrix. Fibers and matrix deform in a single (unified) manner. The fibers follow exactly the deformation of the matrix.

For incompressible materials, we have

$$I_3 - 1 = 0 = \det \underset{\sim}{C} - 1 = 0$$

which may be helpful in solving analytically some problems. It's questionable whether it can be applied to cartilage or cornea, where we may have large changes in volume, due to the repulsive forces exerted between the proteoglycans.

Inextensibility of fibers (if it exists) can be expressed by the condition

$$I_4 - 1 = 0 = \underset{\sim}{C} : \underset{\sim}{M} - 1 = 0 \quad \text{(of the matrix)}$$

In the case of incompressibility or inextensibility of the fibers, we use Lagrange

multipliers in the strain energy density expression

$$\underline{W} \Rightarrow \underline{W} + \frac{1}{2} P_F (I_3 - 1) - \frac{1}{2} T (I_4 - 1)$$

P_F, T: variables to be determined either from boundary conditions or from the equilibrium equations. They are not defined from experiments on the material, although the are present in the constitutive equation of the material.

Again we write

$$I_4 = \underset{\sim}{C} : \underline{M} = |\underset{\sim}{F} \cdot \vec{M}|^2$$

$\vec{n} = \underset{\sim}{F} \cdot \vec{M}$; $\vec{n}$ is not a unit vector itself, it is rather the mapping of the unit vector $\vec{M}$ in the undeformed configuration, to the deformed configuration.

Some kinds of strain energy density function are

- The additive form of $\underline{W}$

$$\underline{W} = \underbrace{\underline{W}^{iso}(I_1, I_2, I_3)}_{\text{bulk or matrix,}} + \underbrace{\underline{W}^{aniso}(I_4, I_4)}_{\text{fibers}}$$

iso: for isotropic, aniso: for anisotropic

We may further split $W^{aniso}(I_4, I_5)$ to

$$\underline{W}^{aniso}(I_4, I_5) = \underline{W}^{aniso}_{(4)}(I_4) + \underline{W}_{(4-5)}(I_4, I_5)$$

where

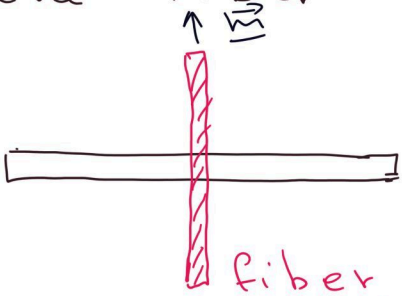
$$W^{aniso}_{(4)}(I_4) = W^{aniso}_{(4)}(\langle I_4 - 1 \rangle^{\alpha})$$

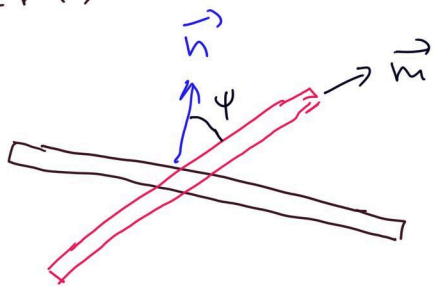
and

$\langle\ \rangle$ indicate the extension of the fibers only. In case of compression this is zero.

$\alpha > 11$ has been used

I_5 used to express the shear between the fiber and the matrix





Undeformed configuration

Deformed configuration

The angle ψ can be written in terms of the invariants as

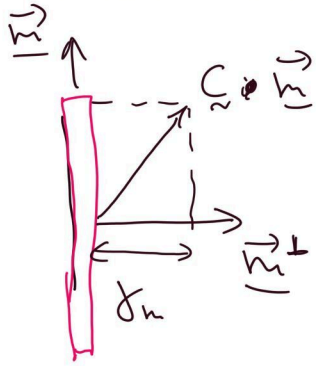
$$\cos\psi = \vec{m}\cdot\vec{n} = \sqrt{\frac{I_3}{I_4\left(I_5 - I_1 I_4 + I_2\right)}}$$

We can write the

$$W^{aniso}_{(4-5)}\left(I_1, I_2, I_3, I_4, I_5\right) = f^{(4)}\left(I_4\right)\tan^2\psi$$

We don't have an additive splitting above. We can have different combinations of interaction

Use of the I_7 invariant to express the fiber-fiber interaction



We may have a deformation described by a vector $\underset{\sim}{C}\cdot\vec{m}$, where

$$\underset{\sim}{C}\cdot\vec{m} = \underbrace{\alpha\,\vec{m}}_{\text{extension along the fiber}} + \underbrace{\gamma_m\,\vec{m}^{\perp}}_{\text{interaction between parallel fiber}}$$

We find that

$$\gamma_m^2 = I_5 - I_4^2$$

In the case of 2 families of parallel fibers, we have clinotropic materials, We use the following invariants

$$\underset{\sim}{I}_1, \underset{\sim}{I}_2, \underset{\sim}{I}_3, \underset{\sim}{I}_4, \underset{\sim}{I}_5, \underset{\sim}{I}_6, \underset{\sim}{I}_7, \underset{\sim}{I}_S$$

$$\underset{\sim}{I}_1 = tr\,\underset{\sim}{C}$$

$$\underset{\sim}{I}_2 = \frac{1}{2}\left(tr^2\underset{\sim}{C} - tr\,\underset{\sim}{C}^2\right)$$

$$\underset{\sim}{I}_3 = \det \underset{\sim}{C}$$

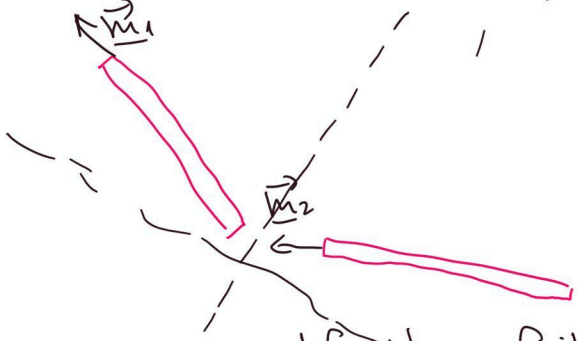
$$\underset{\sim}{I}_4 = \underset{\sim}{C} : \underset{\sim}{M}_2$$

$$\underset{\sim}{I}_5 = \underset{\sim}{C}^2 : \underset{\sim}{M}_1$$

$$\underset{\sim}{I}_6 = \underset{\sim}{C} : \underset{\sim}{M}_2$$

$$\underset{\sim}{I}_7 = \underset{\sim}{C}^2 : \underset{\sim}{M}_2$$

$$\underset{\sim}{I}_S = \underset{\sim}{C} : \underset{\sim}{M}_S$$



S: structure interaction

$$\underset{\sim}{M}_1 = \vec{M}_1 \otimes \vec{M}_1$$

$$\underset{\sim}{M}_2 = \vec{M}_2 \otimes \vec{M}_2$$

If the fibers of the different families, have identical behavior, we have orthotropy along the symmetry directions

$$\underset{\sim}{M}_S = \underset{\sim}{M}_1 \cdot \underset{\sim}{M}_2 + \underset{\sim}{M}_2 \cdot \underset{\sim}{M}_1$$