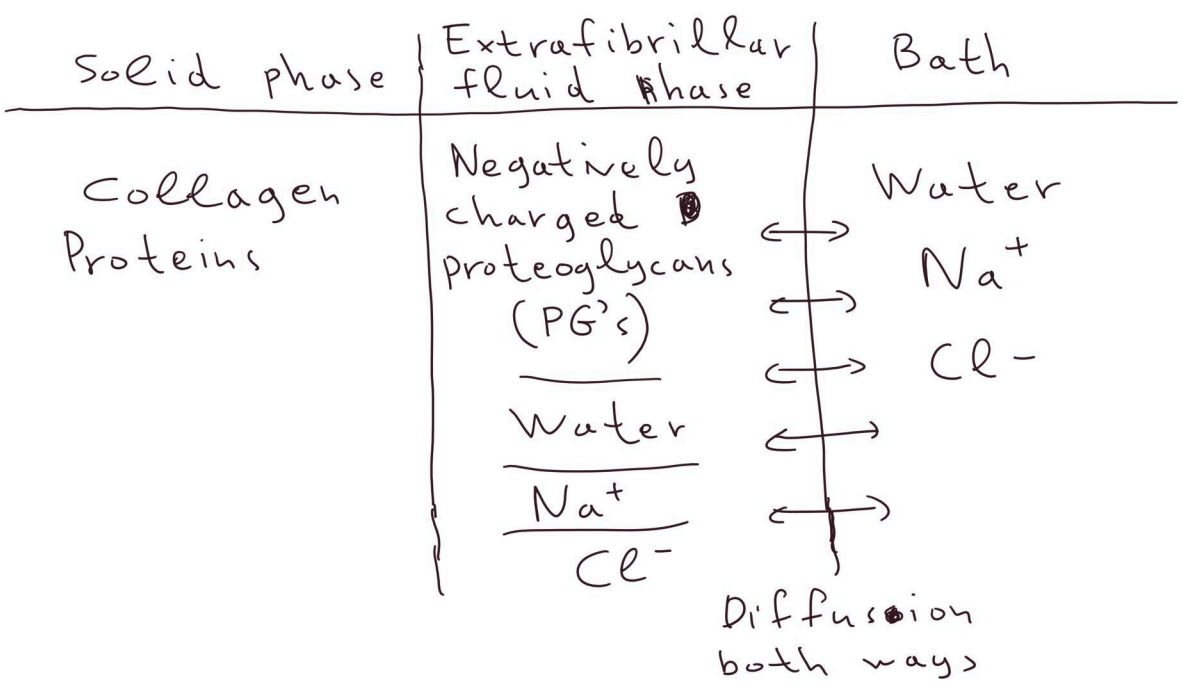
The two phase model for cornea



Solid phase : $S = \{c\}$ collagen

Extrafibrillar fluid phase : $E = \{w, PG, Na, Cl\}$

Apart from mass and momentum balance, we have electroneutrality and incompressibility of all constituents.

The virtual work principle on the medium, in order to find the constitutive relations for the medium, we write

$$\delta W = \underbrace{\underset{\sim}{T} : \delta \underset{\sim}{E}}_{\text{mechanical contribution}} + \underbrace{\sum_{K,k \in K} \psi_{kK}^{ec} \delta m^{kK}}_{\text{electrochemical contribution}}$$

$(\underset{\sim}{T}, \underset{\sim}{E})$: any energetically conjugate pair of stress and strain tensors, e.g

$\left(\begin{matrix} \underset{\sim}{\tau} \\ \underset{\sim}{F} \end{matrix}\right)$: 1^{st} Piola-Kirchhoff stress tenson
deformation gradient tensor

$\left(\begin{matrix} \underset{\approx}{\tau} \\ E_G \end{matrix}\right)$: 2^{nd} Piola-Kirchhoff stress tenson
Green strain tenson

Alternatively the virtual work principle can be written in terms of the mole based potentials g^{ec}_{kK} and moles N_{kK}.

$$\delta \underline{W} = \underset{\sim}{T} : \underset{\sim}{E} + \sum_{k,K \in K} g^{ec}_{kK} N_{kK}$$

The electrochemical potentials g^{ec}_{kK} consist of the following

$$g^{ec}_{kK} = \underbrace{\hat{m}_k \mu^{\circ}_{kK}}_{\text{enthalpy of formation of new mass through eg chemical reaction}} + \underbrace{\hat{v}_k p_{kK}}_{\text{contribution significant for large concentrations of ions}} + \underbrace{RT \ln x_{kK}}_{\text{chemical contribution}} + \underbrace{z_k F \phi_K}_{\text{electrical contribution}}$$

At non-physiological pH, we have for the phases (the extrafibrillar fluid phase)

$$E = \{w, PG, Cl, Na, H, HO\},$$

$$E_{ions} = \{Na, Cl, H, HO\}$$

$$E_{mo} = E - \{PG\} \quad \text{mobile constituents,}$$

we have diffusion of masses

$$\vec{J}_k = \sum_{K \in K} n^{kK} (\vec{v}_{kK} - \vec{v}_s) \qquad \vec{v}_s : \text{velocity of the solid phase}$$

We have trasportation (motion) of electrical charges (i.e we have electrical current).

The electrical current density

$$\vec{\underline{I}}_{ek} = F \sum_{K \in K} z_K \frac{N_{kK}}{V} \vec{v}_{kK}$$

Mass balance equation for the fluid phase

$$\frac{1}{\det \underset{\sim}{F}} \frac{dm^{kE}}{dt} + \operatorname{div} M_{kE} = 0$$

for incompressibility in the whole medium we can write

$$\operatorname{div} \vec{v}_s + \operatorname{div} \vec{J}_E = 0$$

The electroneutrality condition

$$\operatorname{div} \vec{I}_E = 0$$

$\vec{I}_E$: electric current density in the extrafibrillar phase

Momentum balance

$$\operatorname{div} \underset{\sim}{\sigma} + \rho \vec{b} = \vec{0}$$

$\vec{b}$: body forces

We have matrix diffusion equation of the form

$$\vec{j} = -K \vec{f}$$

K: diffusion matrix

$\vec{j}$: diffusion vector $= \begin{bmatrix} \vec{J}_{wE} \\ \vec{J}_{NaE} \\ \vec{J}_{ceE} \end{bmatrix}$

$\vec{f}$: gradient vector $\rightarrow$: $\begin{bmatrix} \vec{\nabla} g^{ec}_{wE} / \hat{v}_w \\ \vec{\nabla} g^{ec}_{NaE} \\ \vec{\nabla} g^{ec}_{ceE} \end{bmatrix}$

Structure of the constitutive equations

$$\delta \underline{W} = \underset{\sim}{T} : \delta \underset{\sim}{E} + \sum_{\ell \in E} g_{\ell E} \delta N^*_{\ell E} + G_L \delta N_{ceE_L} +$$

$$+ P_E \delta I_{inc} + \phi_E \delta I_{ceE}$$

L: Ligands (extensions of collagen fibrils where clorium ions bind)

G_L: affinity potential (chemical affinity)

$\delta \underline{W}$ can be written in a shifted form

as

$$\delta \underline{W} = \overline{\underline{T}} : \delta \underline{E} + \sum_{\ell \in E_{mo}} \bar{g}^{ec}_{\ell E} \delta N^*_{\ell E} + G_L \delta N_{c\ell E_L}$$

where

$$\overline{\underline{T}} = \underline{T} + p_E \det \underline{F} \cdot \underline{F}^{-1} \cdot \underline{F}^{-T}$$

$$\bar{g}^{ec}_{\ell E} = g^{ec}_{\ell E} - \hat{v}_L p_E \,, \quad \ell \in E_{mo}$$

From the virtual work equation we can write the constitutive equation for the porous medium

$$\overline{\underline{T}} = \frac{\partial \underline{W}}{\partial \underline{E}} \,, \quad \bar{g}_{\ell E} = \frac{\partial \underline{W}}{\partial N^*_{\ell E}} \,, \quad G_L = \frac{\partial \underline{W}}{\partial N_{c\ell E_L}}$$

A particular ~~form~~ form of $\underline{W}$ is

$$\underline{W} = \underline{W}_{ch-mech}\left(\underline{E}, N^*_E\right) + \underline{W}_{ch}\left(N^*_E, N_{c\ell E_L}\right) + \\ + \underline{W}_{cf}\left(N_{c\ell E_L}\right)$$

where

$\underline{W}_{ch-mech}$: chemomechanical term (contribution)

$\underline{W}_{ch}$: purely chemical term

$\underline{W}_{cf}$: term related to the binding of cloride ions to the collagen fiber

Another form of chemoelastic energy

$$\underline{W}_{ch-mech}\left(\underline{E}, N^*_E\right) = \underline{T}_0 : \underline{E} + \underline{W}_{ch,1}\left(\underline{E}, N^*_E\right) + \\ + W_{ch,2}\left(N^*_E\right)\left(\underline{W}^{ps}\left(\underline{E}\right) + \underline{W}^{c}\left(\underline{E}\right)\right)$$

where

$\underset{\sim}{T}_0$: initial stress. The term $\underset{\sim}{T}_0 : \underset{\sim}{E}$ refers to the history of the deformation process

$\underline{W}^{gs}(\underset{\sim}{E})$: mechanical energy of the ground substance

$\underline{W}^{c}(\underset{\sim}{E})$: mechanical energy due to the activated collagen fibers

$W_{ch,1}(\underset{\sim}{E}, N_E^*) = -p_{ch}(N_E^*)(\det \underset{\sim}{F} - 1)$: coupled chemomechanical contribution

$W_{ch,2}(N_E^*)$: chemical amplification term

A linear elastic relation can be derived from an energy potential of the form

$$\underline{W}^{gs} = \frac{1}{2}\lambda (\mathrm{tr}\, \underset{\sim}{E}_G)^2 + \mu \underset{\sim}{E}_G : \underset{\sim}{E}_G$$

The 2nd Piola-Kirchhoff stress tensor is given via the relation

$$\underline{\underline{\Sigma}}^{gs} = \frac{\partial \underline{W}^{gs}}{\partial \underset{\sim}{E}_G} = \frac{1}{2}\lambda (\mathrm{tr}\, \underset{\sim}{C} - 3)\underset{\sim}{I} + \mu(\underset{\sim}{C} - \underset{\sim}{I})$$

where $\underset{\sim}{C}$ is the right Cauchy-Green deformation tensor, for which the relation

$$\underset{\sim}{E}_G = \frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I})$$

holds.

The energy potential may lead to a nonlinear stress-strain relation, via the multiplication with a nonlinear function $f(\underset{\sim}{C})$, i.e. we write

$$\underline{W}^{gs} = \underline{W}_0^{gs} f(\underset{\sim}{C})$$

where $\underline{W}_0^{gs}$ is the above mentioned relation for $\underline{W}^{gs}$, that led to the linear stress strain relation.

Transversely isotropic material with a single family of fibers (fibers aligned along one direction)

For isotropic behavior the strain energy density is a function of the 3 invariants of the $\underline{\underline{C}}$ strain tensor, that is

$$\underline{W} = \underline{W}(I_1, I_2, I_3)$$

For transversely isotropic material

$$\underline{W} = \underline{W}(I_1, I_2, I_3, I_4, I_5)$$

is a function of the 3 invariants of $\underline{\underline{C}}$ plus the 2 pseudoinvariants I_4, I_5 of $\underline{\underline{C}}$.