National Technical University of Athens
School of Applied Mathematical
and Physical Sciences
Section of Mechanics

Interdepartmental Program of Postgraduate Studies in "Applied Mechanics",
Erasmus Mundus Joint Master Program "STRAINS" in "Advanced Solid Mechanics",
Final examination in "Biomechanics of Soft Tissues",
Instructor: Assistant Professor D. Eftaxiopoulos,
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Question 1 (5)

1. Prove the relation
$$\int_V \int_{\mathbb{S}^2} \frac{dS}{4\pi} \frac{dV(\mathbf{X})}{V} = 1 \tag{1}$$
where V is the volume of a soft tissue containing collagen fibers as reinforcement, $\mathbb{S}^2$ is the surface of the unit sphere, dS is a differential surface element on the surface of the unit sphere and $\mathbf{X}$ is the position vector of a point of the medium, where the differential volume dV is located at.
2. Prove the following property averaging equation
$$\int_V \int_{\mathbb{S}^2} f_k(\mathcal{V}, \mathbf{m}) \chi_k(\mathbf{X}) \frac{dS}{4\pi} \frac{dV(\mathbf{X})}{V} = n^k \int_{\mathbb{S}^2} f_k(\mathcal{V}, \mathbf{m}) \frac{dS}{4\pi} \tag{2}$$
where $f_k(\mathcal{V}, \mathbf{m})$ is a property depending on a set of variables $\mathcal{V} = \mathcal{V}(\mathbf{X})$ and on the fiber direction $\mathbf{m}$ at point $\mathbf{X}$. $\chi_k(\mathbf{X})$ is a characteristic function which takes the value of 1 at a point $\mathbf{X}$ inside the volume V_k of the species k and is equal to 0 elsewhere. n^k is the volume fraction of the species k in the medium of volume V.
3. If
$$f_k(\mathcal{V}, \mathbf{m}) = \mathbf{m} \otimes \mathbf{m}, \tag{3}$$
i.e the property is direction dependent only, show that
$$\int_V \int_{\mathbb{S}^1} \mathbf{m} \otimes \mathbf{m} \chi_k(\mathbf{X}) \frac{dS}{2\pi} \frac{dV(\mathbf{X})}{V} = n^k \frac{\mathbf{I}}{2} \tag{4}$$
where $\mathbb{S}^1$ is the circumference of the unit disk. Volume V is the volume of a circular cylinder, with the unit disk as cross section and unit axial length. $\mathbf{I}$ is the unit second order tensor.

Question 2 (2)

The linear, chemo - poroelastic equation for small strains, has the form

$$\boldsymbol{\sigma}' = \lambda \mathrm{tr} \boldsymbol{\epsilon}' \mathbf{I} + 2\mu \boldsymbol{\epsilon}' \tag{5}$$

where

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p_w \mathbf{I} \tag{6}$$

is the effective stress, $\boldsymbol{\sigma}$ is the total stress and p_w is the intrinsic fluid pressure. The effective strain is given by the formula

$$\boldsymbol{\epsilon}' = \boldsymbol{\epsilon} + \alpha_c C \mathbf{I} \tag{7}$$

where $\boldsymbol{\epsilon}$ is the total strain tensor, α_c is the coefficient of chemical contraction and C is the concentration of the chemical. When

$$\boldsymbol{\epsilon}' = \mathbf{0} \tag{8}$$

show that:

1. $$\boldsymbol{\sigma}^s = -n^s p_w \mathbf{I} \tag{9}$$
   where $\boldsymbol{\sigma}^s$ is the stress tensor of the solid skeleton and n^s is the volume fraction of the solid phase.
2. Uniform contraction takes place in the tissue, due to the increase of the concentration C of the chemical.

Question 3 (3)
The chemoelastic energy density for cartilage, per unit volume in the undeforrmed configuration, is given via the formula

$$\underline{W}_{\text{ch}-\text{mech}}(\boldsymbol{\mathcal{E}}) = -p_{\text{ch}}(\boldsymbol{\mathcal{E}})\epsilon_{\text{ch}} + \underline{W}_{\text{mech}}(\mathbf{E}) + \int_0^{m^{wI}} P_{\text{adh}}(m)\frac{dm}{\rho_w} \tag{10}$$

where

1. -$p_{\text{ch}}(\boldsymbol{\mathcal{E}})\epsilon_{\text{ch}}$ is a coupled chemomechanical energy density term, depending on the chemical pressure $p_{\text{ch}}(\boldsymbol{\mathcal{E}})$ and on the chemical strain scalar
   $$\epsilon_{\text{ch}} = \det\mathbf{F} - 1. \tag{11}$$
   p_{ch} in turn, depends on strain through the scalar ϵ_{ch}. $\boldsymbol{\mathcal{E}}$ is the generalised strain tensor and $\mathbf{F}$ is the deformation gradient tensor.
2. $\underline{W}_{\text{mech}}(\mathbf{E})$ is a purely mechanical energy density term, depending on the Lagrangian strain tensor $\mathbf{E}$.
3. $\displaystyle\int_0^{m^{wI}} P_{\text{adh}}(m)\frac{dm}{\rho_w}$ is a purely chemical energy density term, depending on the mass m^{wI} of the intrafibrillar water. $P_{\text{adh}}(m)$ is a constitutive function representing the contribution to the adhesion pressure p_{adh} in the intrafibrillar compartment. ρ_w is the mass density of the water.

Show that the chemical stress $\mathbf{T}^{\text{ch}}$ emerging from the chemomechanical term - $p_{\text{ch}}(\boldsymbol{\mathcal{E}})\epsilon_{\text{ch}}$, takes the form

$$\mathbf{T}^{\text{ch}} = -\left(p_{\text{ch}} + \epsilon_{\text{ch}}\frac{\partial p_{\text{ch}}}{\partial \epsilon_{\text{ch}}}\right)(\det\mathbf{F})\ \mathbf{F}^{-1}\cdot\mathbf{F}^{-\text{T}}. \tag{12}$$