**Πρόταση:** Έστω $I \subset \mathbb{R}$ διάστημα, $f: I \longrightarrow \mathbb{R}$ παραγωγίσιμη με $f(x) \neq 0$, $\forall x \in I$. Τότε,
$$\int \frac{f'(x)}{f(x)}\,dx = \ln(|f(x)|) + c, \quad \forall x \in I, \ c \in \mathbb{R}.$$

**Απόδειξη:** Η ισότητα εκφράζει το γεγονός ότι η παράγωγος της $\ln(|f(x)|)$ ισούται με $\frac{f'(x)}{f(x)}$, $\forall x \in I$. Πράγματι, αφού $f(x) \neq 0$, $\forall x \in I$, και $f$ συνεχής στο $I$, ως παραγωγίσιμη, η $f$ θα διατηρεί πρόσημο στο διάστημα $I$.
Αν $f(x) > 0$, $\forall x \in I$, τότε $\ln(|f(x)|) = \ln(f(x))$, $\forall x \in I \Rightarrow \frac{d}{dx}\left[\ln(|f(x)|)\right] = \frac{f'(x)}{f(x)}$, $\forall x \in I$. Αν $f(x) < 0$, $\forall x \in I$, τότε $\ln|f(x)| = \ln(-f(x))$, $\forall x \in I$. Άρα,
$$\frac{d}{dx}\left[\ln|f(x)|\right] = \frac{d}{dx}\left[\ln(-f(x))\right] = \frac{-f'(x)}{-f(x)} = \frac{f'(x)}{f(x)}, \quad \forall x \in I.$$

**Παραδείγματα:** 1)
$$\int \frac{2x+3}{x^2+3x+1}\,dx = \int \frac{(x^2+3x+1)'}{x^2+3x+1}\,dx = \ln|x^2+3x+1| + c, \quad \forall x \in I,$$
όπου $I$ διάστημα στο οποίο $x^2+3x+1 \neq 0$, $\forall x \in I$. Το τριώνυμο $x^2+3x+1$ έχει ρίζες τους $\frac{-3-\sqrt{5}}{2}$ και $\frac{-3+\sqrt{5}}{2}$. Μπορούμε λοιπόν να πάρουμε σαν $I$ ένα από τα $\left(-\infty, \frac{-3-\sqrt{5}}{2}\right)$, $\left(\frac{-3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}\right)$, $\left(\frac{-3+\sqrt{5}}{2}, +\infty\right)$.
Στο πρώτο και στο τρίτο διάστημα έχουμε $x^2+3x+1 > 0$, στο μεσαίο $x^2+3x+1 < 0$.

2)
$$\int \frac{3x+4}{x^2+1}\,dx = \int \frac{3x}{x^2+1}\,dx + \int \frac{4}{x^2+1}\,dx = \frac{3}{2}\int \frac{2x}{x^2+1}\,dx + 4\int \frac{dx}{x^2+1} =$$
$$= \frac{3}{2}\ln(x^2+1) + 4\,\mathrm{Arctan}(x) + c, \quad \forall x \in \mathbb{R}$$

3)
$$\int \frac{4x+3}{(4x^2+6x+7)^2}\,dx \underset{du=(8x+6)dx}{\overset{u=4x^2+6x+7}{=}} \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} + c = -\frac{1}{2(4x^2+6x+7)} + c$$

4) $\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{(\cos(x))'}{\cos(x)}dx = -\ln|\cos(x)| + c$, $x\in I$, για κάθε διάστημα $I \subset \mathbb{R}$ που δεν περιέχει ρίζες του cos.

5) $\int \cot(x)dx = \int \frac{\cos(x)}{\sin(x)}dx = \int \frac{(\sin(x))'}{\sin(x)}dx = \ln|\sin(x)| + c$, $x\in I$, $I\subset\mathbb{R}$ διάστημα που δεν περιέχει ρίζες του sin.

6) $\int \frac{e^x+x}{2e^x+x^2}dx = \frac{1}{2}\int \frac{2(e^x+x)}{2e^x+x^2}dx = \frac{1}{2}\int \frac{(2e^x+x^2)'}{2e^x+x^2}dx = \frac{1}{2}\ln(2e^x+x^2)+c$, $x\in\mathbb{R}$.

7) $\int \frac{dx}{x\ln(x)} = \int \frac{1/x}{\ln(x)}dx = \int \frac{(\ln(x))'}{\ln(x)}dx = \ln|\ln(x)| + c$, $x\in I$, ώστε $I\subset(0,+\infty)$ διάστημα με $1\notin I$.

8) $\int \frac{e^x-3}{e^x+5}dx = \int \frac{e^x+5-8}{e^x+5}dx = \int\left(1-\frac{8}{e^x+5}\right)dx = \int dx - 8\int\frac{dx}{e^x+5} =$

$\xrightarrow[\substack{du=e^x dx=\\=(u-5)dx}]{u=e^x+5} x - 8\int\frac{du}{u(u-5)} = x - 8\int\frac{1}{5}\left(\frac{1}{u-5}-\frac{1}{u}\right)du = x - \frac{8}{5}\int\frac{du}{u-5} + \frac{8}{5}\int\frac{du}{u} =$

$= x - \frac{8}{5}\ln|u-5| + \frac{8}{5}\ln|u| + c = x - \frac{8}{5}\ln(e^x) + \frac{8}{5}\ln(e^x+5) + c =$

$= x - \frac{8x}{5} + \frac{8}{5}\ln(e^x+5) + c = -\frac{3x}{5} + \frac{8}{5}\ln(e^x+5)+c$, $x\in\mathbb{R}$.

9) $\int \frac{e^x}{e^x+7}dx = \int\frac{(e^x+7)'}{e^x+7}dx = \ln(e^x+7)+c$, $x\in\mathbb{R}$.

10) $\int\frac{\sin(2x)}{\sin^2(x)}dx = \int\frac{2\sin(x)\cos(x)}{\sin^2(x)}dx = 2\int\frac{\cos(x)}{\sin(x)}dx = 2\ln|\sin(x)|+c$, $x\in I\subset\mathbb{R}$ διάστημα

$I$ δεν περιέχει ρίζες του sin.

11) $\int x\tan(x^2)dx = \int\frac{x\sin(x^2)}{\cos(x^2)}dx = -\frac{1}{2}\int\frac{-2x\sin(x^2)}{\cos(x^2)}dx =$

$= -\frac{1}{2}\int\frac{(\cos(x^2))'}{\cos(x^2)}dx = -\frac{1}{2}\ln|\cos(x^2)|+c$, $x\in I\subset\mathbb{R}$ διάστημα που δεν περιέχει $\pm$ τετραγωνική ρίζα ρίζας του cos.

12) $\int \frac{x-4}{x^2-2x+5}dx = \frac{1}{2}\int \frac{2x-8}{x^2-2x+5}dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}dx + \frac{1}{2}\int \frac{-6}{x^2-2x+5}dx =$

$= \frac{1}{2}\int \frac{(x^2-2x+5)'}{x^2-2x+5}dx - 3\int \frac{dx}{x^2-2x+4+1} = \frac{1}{2}\ln(x^2-2x+5) - 3\int \frac{dx}{(x-1)^2+4} =$

$= \frac{1}{2}\ln(x^2-2x+5) - \frac{3}{4}\int \frac{dx}{\left(\frac{x-1}{2}\right)^2+1} = \frac{1}{2}\ln(x^2-2x+5) - \frac{3}{4}\cdot 2\,\mathrm{Arctan}\left(\frac{x-1}{2}\right) + c$

$= \frac{1}{2}\ln(x^2-2x+5) - \frac{3}{2}\mathrm{Arctan}\left(\frac{x-1}{2}\right) + c \quad , x\in\mathbb{R}.$

Παρατήρηση: $x^2-2x+5 = (x-1)^2+4>0, \ \forall x\in\mathbb{R}$.

Παραγοντική Ολοκλήρωση (ολοκλήρωση κατά μέρη).

Πρόταση: Αν $I\subset\mathbb{R}$ διάστημα, $f: I\to\mathbb{R}$ και $g: I\to\mathbb{R}$ παραγωγίσιμες τότε $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx \quad , \forall x\in I$

Απόδειξη: Κατ' αρχήν η παραπάνω ισότητα έχει την εξής ερμηνεία:

Η $f'g$ έχει αντιπαράγωγο στο $I \iff$ Η $fg'$ έχει αντιπαράγωγο στο $I$

Επιπλέον, η $F(x)$ είναι αντιπαράγωγος της $fg'$ στο $I \iff fg-F$ είναι αντιπαράγωγος της $f'g$ στο $I$

Πράγματι, $(fg-F)' = f'g \iff (fg)'-F' = f'g \iff f'g+fg'-F' = f'g \iff$

$\iff fg' = F'$.

Παραδείγματα: 1) $\int xe^x dx = \int x(e^x)'dx = xe^x - \int x'e^x dx = xe^x - \int e^x dx =$

$= xe^x - e^x + c \quad , x\in\mathbb{R}$

2) $\int x^2 e^{3x} dx = \int x^2 \left(\frac{1}{3}e^{3x}\right)' dx = \frac{x^2 e^{3x}}{3} - \int 2x \frac{1}{3} e^{3x} dx =$

$= \frac{x^2 e^{3x}}{3} - \frac{2}{3} \int x \left(\frac{1}{3} e^{3x}\right)' dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left[\frac{x}{3} e^{3x} - \int \frac{1}{3} e^{3x} dx\right] =$

$= \frac{x^2 e^{3x}}{3} - \frac{2x}{9} e^{3x} + \frac{2}{9} \int e^{3x} dx = \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) e^{3x} + C, \quad x \in \mathbb{R}$

3) $\int e^x \cos(x) dx = \int (e^x)' \cos(x) dx = e^x \cos(x) - \int e^x (-\sin(x)) dx = e^x \cos(x) + \int e^x \sin(x) dx =$

$= e^x \cos(x) + \int (e^x)' \sin(x) dx = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) dx \Rightarrow$

$\Rightarrow 2\int e^x \cos(x) dx = e^x \cos(x) + e^x \sin(x) + C \Rightarrow \int e^x \cos(x) dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + C$

$(x \in \mathbb{R})$

4) $\int x e^x \cos(x) dx = \int (xe^x - e^x)' \cos(x) dx = (xe^x - e^x)\cos(x) + \int (xe^x - e^x)\sin(x) dx =$

$= (xe^x - e^x)\cos(x) + \int (xe^x - e^x - e^x)' \sin(x) dx = (x-1)e^x \cos(x) + (x-2)e^x \sin(x) - \int (xe^x - 2e^x)\cos(x) dx$

$= (x-1)e^x \cos(x) + (x-2)e^x \sin(x) - \int (xe^x - 2e^x)\cos(x) dx =$

$= (x-1)e^x \cos(x) + (x-2)e^x \sin(x) - \int x e^x \cos(x) dx + 2\int e^x \cos(x) dx \Rightarrow$

$\Rightarrow 2\int x e^x \cos(x) dx \overset{3)}{=} (x-1)e^x \cos(x) + (x-2)e^x \sin(x) + e^x \cos(x) + e^x \sin(x) + C$

$\Rightarrow \int x e^x \cos(x) dx = \frac{x}{2} e^x \cos(x) + \frac{x-1}{2} e^x \sin(x) + C, \quad x \in \mathbb{R}.$

5) $\int \ln(x) dx = \int (x)' \ln(x) dx = x \ln(x) - \int x \cdot \frac{1}{x} dx = x\ln(x) - x + C, \quad x > 0$

6) $\int \sqrt{x} \ln(x) dx = \int \left(\frac{2}{3} x^{3/2}\right)' \ln(x) dx = \frac{2}{3} x^{3/2} \ln(x) - \int \frac{2}{3} x^{3/2} \cdot \frac{1}{x} dx =$

$= \frac{2}{3} x^{3/2} \ln(x) - \frac{2}{3}\int \sqrt{x} dx = \frac{2}{3} x^{3/2} \ln(x) - \frac{2}{3} \cdot \frac{2}{3} x^{3/2} + C = \frac{2}{3} x^{3/2} \ln(x) - \frac{4}{9} x^{3/2} + C, (x>0)$

7) $\int \frac{\ln(x)}{x} dx = \int (\ln(x))' \ln(x) dx = [\ln(x)]^2 - \int \ln(x) \cdot \frac{1}{x} dx \Rightarrow 2\int \frac{\ln(x)}{x} dx = [\ln(x)]^2 + C \Rightarrow$

$\Rightarrow \int \frac{\ln(x)}{x} dx = \frac{1}{2}[\ln(x)]^2 + C, \quad x > 0.$

8) $\int \cos[\ln(x)]\,dx = \int (x)'\cos[\ln(x)]\,dx = x\cos[\ln(x)] - \int x\cdot(-1)\sin[\ln(x)]\cdot\frac{1}{x}\,dx =$

$= x\cos[\ln(x)] + \int \sin[\ln(x)]\,dx = x\cos[\ln(x)] + \int (x)'\sin[\ln(x)]\,dx =$

$= x\cos[\ln(x)] + x\sin[\ln(x)] - \int x\cdot\cos[\ln(x)]\cdot\frac{1}{x}\,dx = x\cos[\ln(x)] + x\sin[\ln(x)] - \int\cos[\ln(x)]\,dx$

$= x\left[\cos[\ln(x)] + \sin[\ln(x)]\right] - \int \cos[\ln(x)]\,dx \Rightarrow 2\int\cos[\ln(x)]\,dx = x\left[\cos[\ln(x)] + \sin[\ln(x)]\right] + C$

$\Rightarrow \int \cos[\ln(x)]\,dx = x\left[\cos[\ln(x)] + \sin[\ln(x)]\right] + C \quad , \quad x > 0$

9) $\int \mathrm{Arctan}(x)\,dx = \int (x)'\,\mathrm{Arctan}(x)\,dx = x\,\mathrm{Arctan}(x) - \int x\left[\mathrm{Arctan}(x)\right]'\,dx =$

$= x\,\mathrm{Arctan}(x) - \int \frac{x}{1+x^2}\,dx = x\,\mathrm{Arctan}(x) - \frac{1}{2}\int\frac{2x}{1+x^2}\,dx = x\,\mathrm{Arctan}(x) - \frac{1}{2}\ln(1+x^2) + C$

$(x\in\mathbb{R})$

10) $\int x\,\mathrm{Arctan}(x)\,dx = \int\left(\frac{x^2}{2}\right)'\mathrm{Arctan}(x)\,dx = \frac{x^2}{2}\mathrm{Arctan}(x) - \int\frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx = \frac{x^2}{2}\mathrm{Arctan}(x) -$

$-\frac{1}{2}\int\frac{x^2+1-1}{x^2+1}\,dx = \frac{x^2}{2}\mathrm{Arctan}(x) - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx = \frac{x^2}{2}\mathrm{Arctan}(x) - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{1+x^2} =$

$= \frac{x^2}{2}\mathrm{Arctan}(x) - \frac{1}{2}x + \frac{1}{2}\mathrm{Arctan}(x) + C \quad , \quad x\in\mathbb{R}.$

11) $\int \mathrm{Arcsin}(x)\,dx = \int (x)'\,\mathrm{Arcsin}(x)\,dx = x\,\mathrm{Arcsin}(x) - \int x\cdot\left(\mathrm{Arcsin}(x)\right)'dx = x\,\mathrm{Arcsin}(x) - \int\frac{x}{\sqrt{1-x^2}}\,dx =$

$\overset{1-x^2=u}{\underset{-2x\,dx=du}{=}} x\,\mathrm{Arcsin}(x) + \frac{1}{2}\int\frac{du}{\sqrt{u}} = x\,\mathrm{Arcsin}(x) + \sqrt{u} + C = x\,\mathrm{Arcsin}(x) + \sqrt{1-x^2} + C \quad , \quad -1 < x < 1$

12) $\int x\sin(x)\,dx = \int x\left(-\cos(x)\right)'dx = -x\cos(x) - \int (x)'\left(-\cos(x)\right)dx = -x\cos(x) + \int\cos(x)\,dx =$

$= -x\cos(x) + \sin(x) + C \quad , \quad x\in\mathbb{R}$

13) $\int x^2\sin(x)\,dx = \int x^2\left(-\cos(x)\right)'dx = -x^2\cos(x) - \int 2x\left(-\cos(x)\right)dx = -x^2\cos(x) + 2\int x\cos(x)\,dx$

$= -x^2\cos(x) + 2\int x\left(\sin(x)\right)'dx = -x^2\cos(x) + 2x\sin(x) - 2\int\sin(x)\,dx =$

$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C \quad , \quad x\in\mathbb{R}.$

14) $\int x^2\sin^2(x)dx = \int x^2\,\frac{1-\cos(2x)}{2}dx = \int\frac{x^2}{2}dx - \frac{1}{2}\int x^2\cos(2x)dx =$

$= \frac{x^3}{6} - \frac{1}{2}\int x^2\cos(2x)dx$. Υπολογίζουμε το $\int x^2\cos(2x)dx = \int x^2\left(\frac{1}{2}\sin(2x)\right)'dx =$

$= \frac{x^2}{2}\sin(2x) - \int 2x\cdot\frac{1}{2}\sin(2x)dx = \frac{x^2}{2}\sin(2x) - \int x\sin(2x)dx = \frac{x^2}{2}\sin(2x) + \int x\left(\frac{1}{2}\cos(2x)\right)'dx$

$= \frac{x^2}{2}\sin(2x) + \frac{x}{2}\cos(2x) - \int\frac{1}{2}\cos(2x)dx = \frac{x^2}{2}\sin(2x) + \frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C$

Τελικά, $\int x^2\sin^2(x)dx = \frac{x^3}{6} - \frac{1}{4}x^2\sin(2x) - \frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x) + C$, $x\in\mathbb{R}$.

15) $\int\sin(\sqrt{x})dx \underset{\substack{dt=\frac{dx}{2\sqrt{x}} \\ dx=2t\,dt}}{\overset{t=\sqrt{x}}{=}} \int\sin(t)\,2t\,dt = 2\int t\sin(t)dt \overset{(12)}{=} 2\left[-t\cos(t)+\sin(t)\right] + C =$

$= -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C$, $(x>0)$.

16) $\int\frac{\text{Arcsin}(x)}{\sqrt{1-x}}dx = \int\left[(-2)\sqrt{1-x}\right]'\text{Arcsin}(x)dx = (-2)\sqrt{1-x}\,\text{Arcsin}(x) - \int(-2)\sqrt{1-x}\,\frac{1}{\sqrt{1-x^2}}dx =$

$= -2\sqrt{1-x}\,\text{Arcsin}(x) + 2\int\frac{\sqrt{1-x}}{\sqrt{1-x}\,\sqrt{1+x}}dx = -2\sqrt{1-x}\,\text{Arcsin}(x) + 2\int\frac{dx}{\sqrt{1+x}} =$

$= -2\sqrt{1-x}\,\text{Arcsin}(x) + 4\sqrt{1+x} + C$, $-1<x<1$.

17) $\int\sqrt{1-x^2}dx = \int(x)'\sqrt{1-x^2}dx = x\sqrt{1-x^2} - \int x\left(\sqrt{1-x^2}\right)'dx = x\sqrt{1-x^2} - \int x\cdot\frac{-2x}{2\sqrt{1-x^2}}dx =$

$= x\sqrt{1-x^2} + \int\frac{x^2}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} + \int\frac{x^2-1+1}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} - \int\frac{1-x^2}{\sqrt{1-x^2}}dx + \int\frac{dx}{\sqrt{1-x^2}}$

$= x\sqrt{1-x^2} - \int\sqrt{1-x^2}dx + \text{Arcsin}(x) \Rightarrow 2\int\sqrt{1-x^2}dx = x\sqrt{1-x^2} + \text{Arcsin}(x) + C$

$\Rightarrow \int\sqrt{1-x^2}dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\text{Arcsin}(x) + C$, $-1<x<1$

18) $\int\frac{x^2}{\sqrt{1-x^2}}dx = \int\frac{x^2-1+1}{\sqrt{1-x^2}}dx = \int\frac{dx}{\sqrt{1-x^2}} - \int\sqrt{1-x^2}dx \overset{17)}{=} \text{Arcsin}(x) - \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\text{Arcsin}(x) + C$

$= \frac{1}{2}\text{Arcsin}(x) - \frac{1}{2}x\sqrt{1-x^2} + C$, $-1<x<1$

19) $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int x\,\frac{x^2}{\sqrt{1-x^2}}\,dx = \int x\cdot\left[\frac{1}{2}\text{Arcsin}(x) - \frac{1}{2}x\sqrt{1-x^2}\right]'dx$, αφού 18).

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^2}}\,dx = x\left[\frac{1}{2}\text{Arcsin}(x) - \frac{1}{2}x\sqrt{1-x^2}\right] - \int\left[\frac{1}{2}\text{Arcsin}(x) - \frac{1}{2}x\sqrt{1-x^2}\right]dx =$$

$$= \frac{x}{2}\text{Arcsin}(x) - \frac{x^2}{2}\sqrt{1-x^2} + \frac{1}{2}\int x\sqrt{1-x^2}\,dx - \frac{1}{2}\int \text{Arcsin}(x)\,dx =$$

$$= \frac{x}{2}\text{Arcsin}(x) - \frac{x^2}{2}\sqrt{1-x^2} - \frac{1}{2}\left[x\,\text{Arcsin}(x) + \sqrt{1-x^2}\right] + \frac{1}{2}\int x\sqrt{1-x^2}\,dx, \text{ αφού 11).}$$

και $\displaystyle\int x\sqrt{1-x^2}\,dx \overset{u=1-x^2}{\underset{du=-2x\,dx}{=}} -\frac{1}{2}\int\sqrt{u}\,du = -\frac{1}{2}\,\frac{2}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C, \quad -1<x<1.$

Άρα, $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{x^2}{2}\sqrt{1-x^2} - \frac{1}{2}\sqrt{1-x^2} - \frac{1}{6}(1-x^2)^{3/2} + C, \quad -1<x<1.$

20) $\displaystyle\int (1-x^2)^{3/2}dx = \int(1-x^2)\sqrt{1-x^2}\,dx = \int\sqrt{1-x^2}\,dx - \int x^2\sqrt{1-x^2}\,dx = \int\sqrt{1-x^2}\,dx - \int x\left[x\sqrt{1-x^2}\right]dx =$

$$= \int\sqrt{1-x^2}\,dx - \int x\left[-\frac{1}{3}(1-x^2)^{3/2}\right]'dx =$$

$$= \int\sqrt{1-x^2}\,dx + \int x\left[\frac{1}{3}(1-x^2)^{3/2}\right]'dx = \int\sqrt{1-x^2}\,dx + x\,\frac{1}{3}(1-x^2)^{3/2} - \int\frac{1}{3}(1-x^2)^{3/2}dx.$$

$$\Rightarrow \frac{4}{3}\int(1-x^2)^{3/2}dx = \frac{x}{3}(1-x^2)^{3/2} + \int\sqrt{1-x^2}\,dx \Rightarrow \int(1-x^2)^{3/2}dx = \frac{1}{4}x(1-x^2)^{3/2} +$$

$$+ \frac{3}{4}\int\sqrt{1-x^2}\,dx \overset{(17)}{=} \frac{x}{4}(1-x^2)^{3/2} + \frac{3}{4}\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\text{Arcsin}(x)\right] + C \Rightarrow$$

$$\Rightarrow \int(1-x^2)^{3/2}dx = \frac{x}{4}(1-x^2)^{3/2} + \frac{3x}{8}\sqrt{1-x^2} + \frac{3}{8}\text{Arcsin}(x) + C, \quad -1<x<1.$$

21) $\displaystyle\int x^2\sqrt{1-x^2}\,dx = \int x\cdot x\sqrt{1-x^2}\,dx = \int x\cdot\left[-\frac{1}{3}(1-x^2)^{3/2}\right]'dx = \frac{-x}{3}(1-x^2)^{3/2} + \frac{1}{3}\int(1-x^2)^{3/2}dx =$

$$\overset{(20)}{=} -\frac{1}{4}x(1-x^2)^{3/2} + \frac{1}{8}x\sqrt{1-x^2} + \frac{1}{8}\text{Arcsin}(x) + C, \quad -1<x<1.$$

22) $\displaystyle\int\frac{dx}{(1+x^2)^2} = \int\frac{1+x^2-x^2}{(1+x^2)^2}dx = \int\left[\frac{1+x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2}\right]dx = \int\frac{dx}{1+x^2} + \int\frac{-x^2}{(1+x^2)^2}dx = \text{Arctan}(x) + \int\frac{-x\cdot x}{(1+x^2)^2}dx$

$$= \text{Arctan}(x) + \int x\left[\frac{1}{2}\,\frac{1}{1+x^2}\right]'dx = \text{Arctan}(x) + \frac{x}{2(1+x^2)} - \int\frac{1}{2}\,\frac{1}{1+x^2}\,dx =$$

$$= \frac{1}{2}\text{Arctan}(x) + \frac{x}{2(1+x^2)} + C, \quad x\in\mathbb{R}.$$