NATIONAL TECHNICAL UNIVERSITY OF ATHENS

School of Civil Engineering – Geotechnical Department

Computational Methods in the Analysis of Underground Structures

Spring Term 2023 – 24

Lecture Series in Postgraduate Programs:

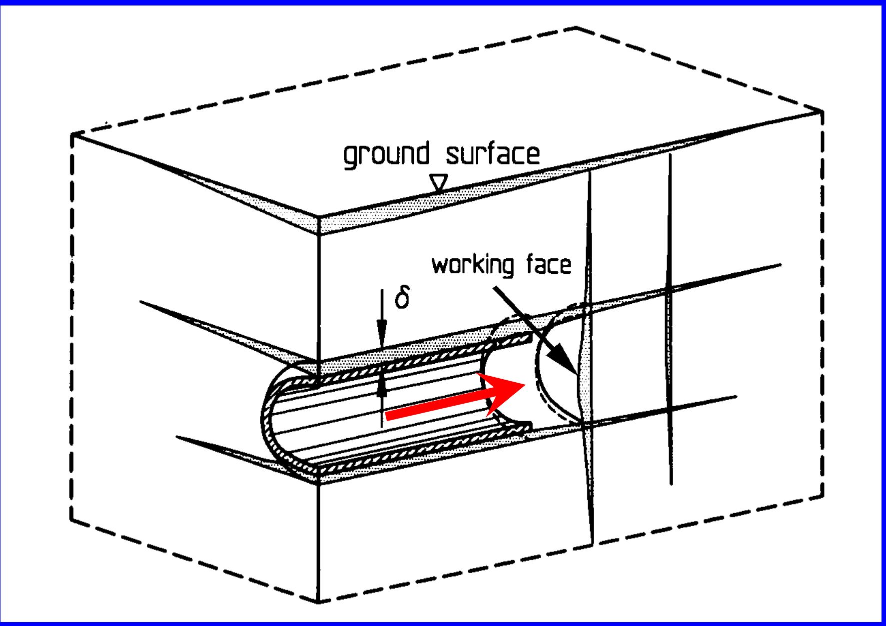
1. Analysis and Design of Structures (DSAK)
2. Design and Construction of Underground Structures (SKYE)

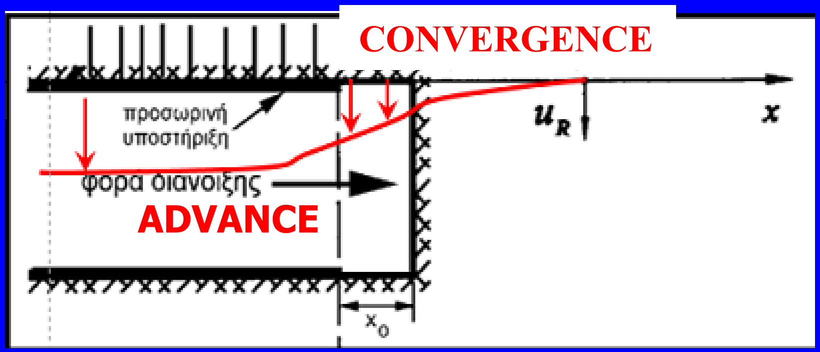
Instructor: Michael Kavvadas, Emer. Professor NTUA

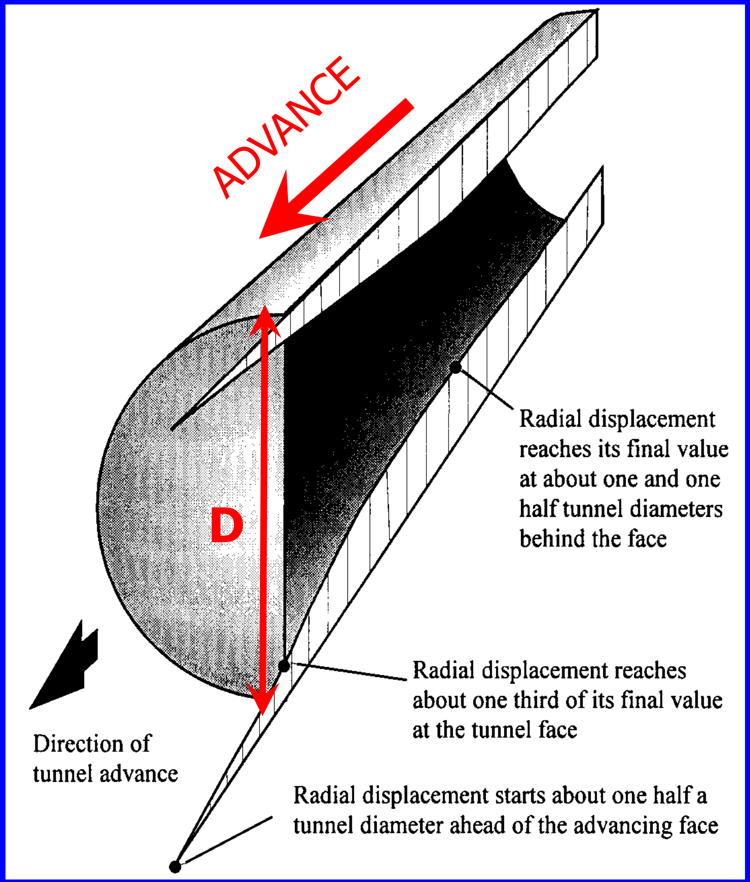
LECTURE 2: Stresses and deformations around a cylindrical tunnel (2D elasto-plastic analysis)

06.03.2024

Evolution of wall convergence along the tunnel axis (x)







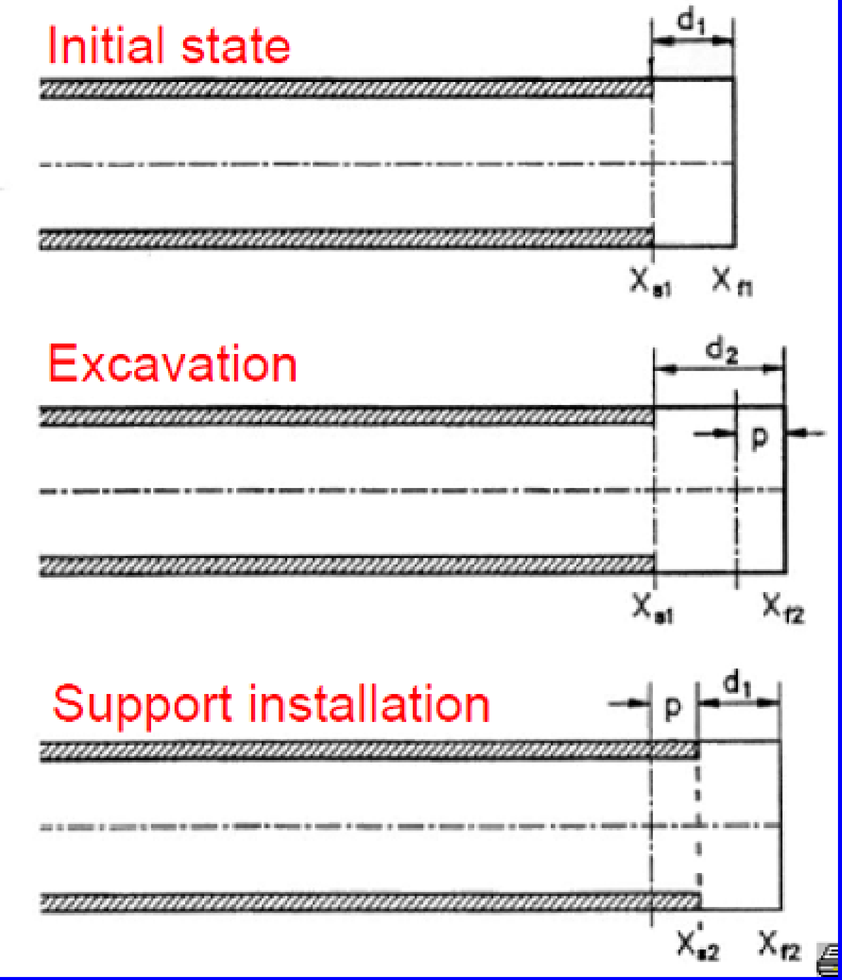
NOTE: Floor rise is equal to crest settlement

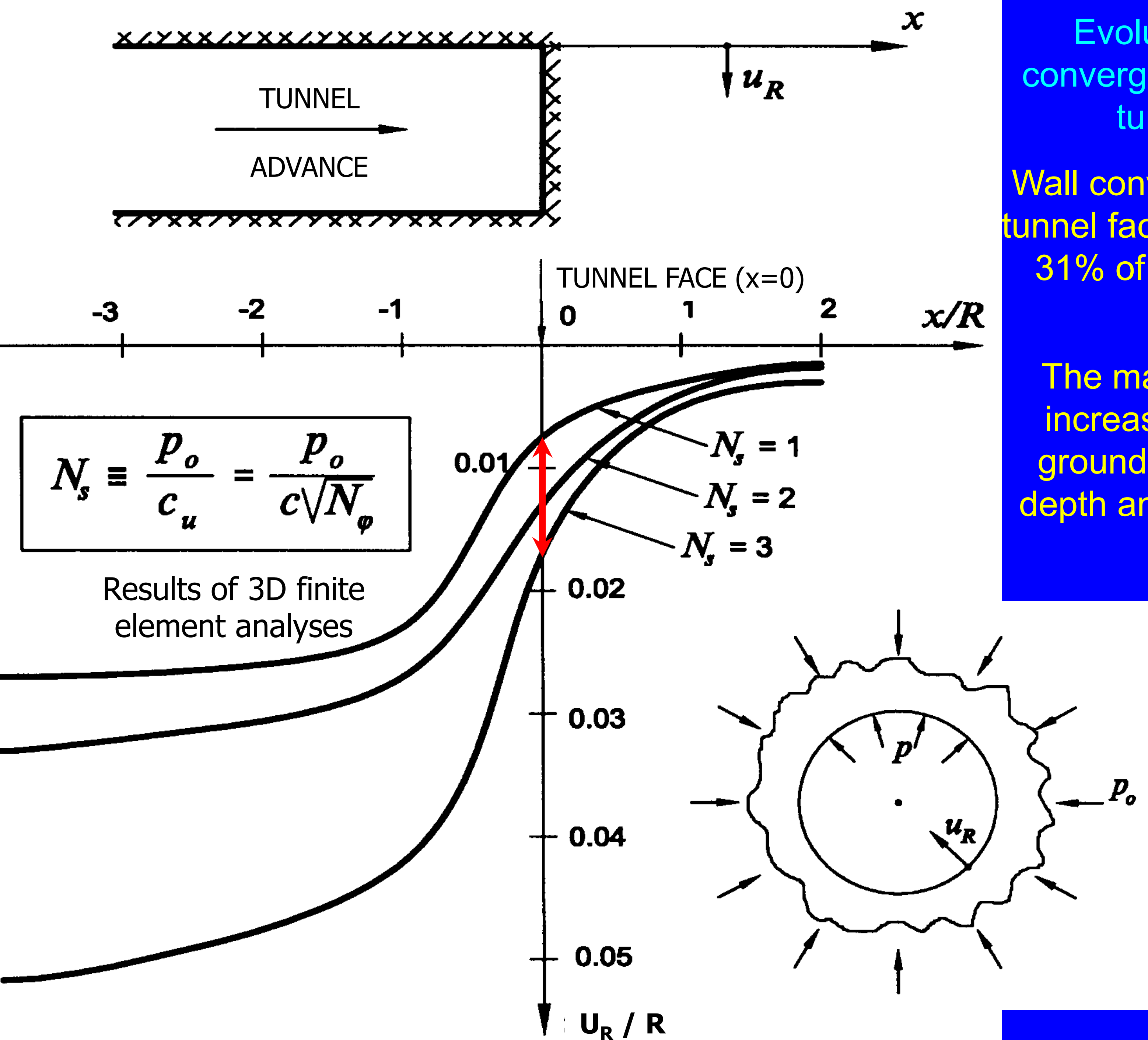
- Convergence starts at distance 0.5-0.75 D ahead of the tunnel face
- 30% - 50% of the total convergence has occurred at the tunnel face
- Wall convergence ceases to increase beyond about 1.5 D behind the tunnel face

Evolution of wall convergence along the tunnel axis (x)

Tunnel advance and wall support in steps with length (p).

The front part of the tunnel, close to the tunnel face (length d_1), remains unsupported for construction purposes (access limitation of machinery). The maximum unsupported length close to the tunnel face is $d_2 = d_1 + p$





Evolution of wall convergence along the tunnel axis

Wall convergence at the tunnel face (x=0) is about 31% of the maximum value

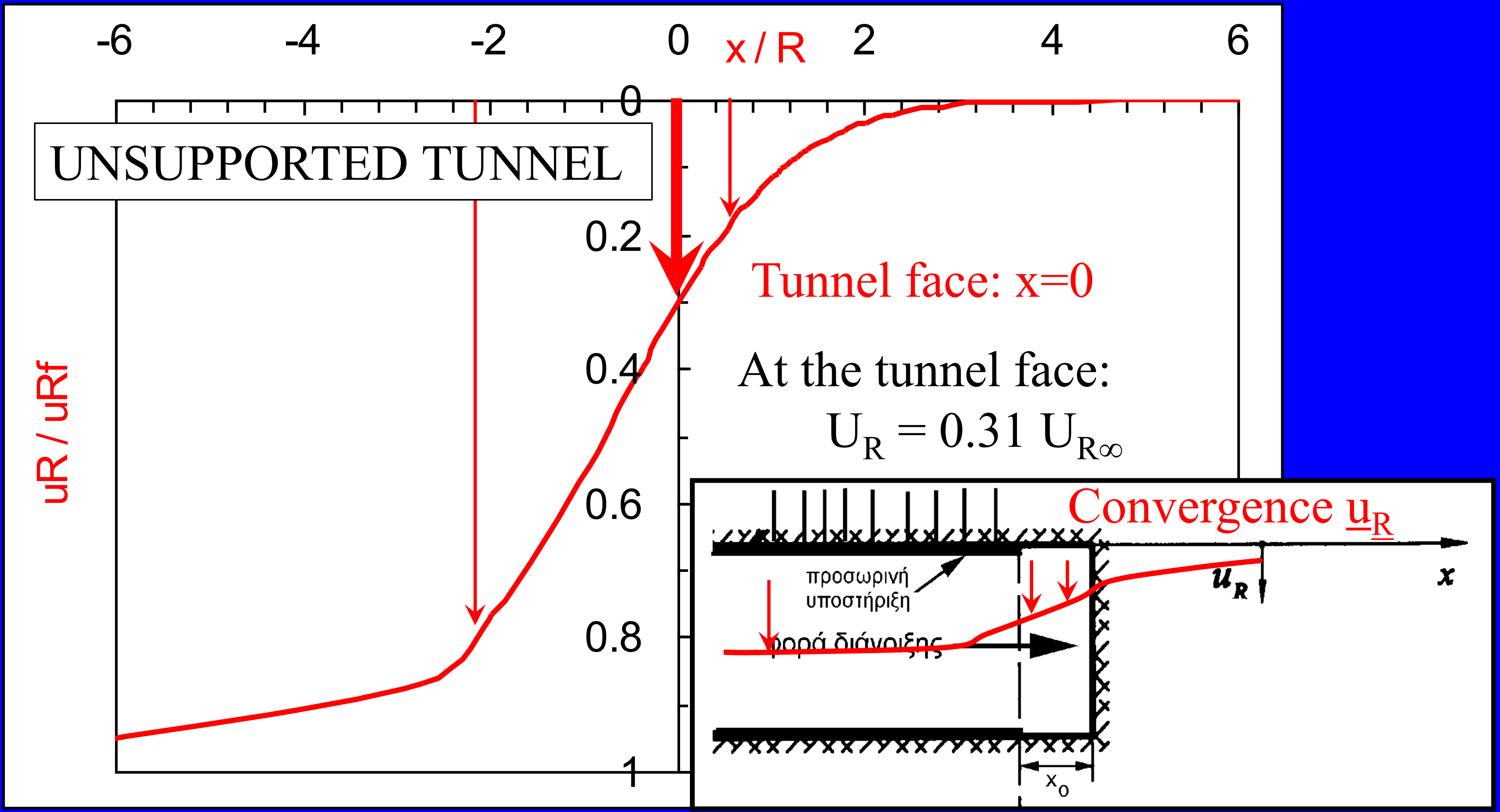
The maximum value increases in weaker ground, larger tunnel depth and larger tunnel size.

Evolution of wall convergence along the tunnel axis (x)

Reference	Analytical Solution	Medium Behaviour
Pane and Guenot (1982)	$\frac{u_r}{u_{max}} = 0.28 + 0.72[1 - (\frac{0.84}{0.84 + {}^x/_R})^2$	Elasto-Plastic
Corbeta et al. (1991)	$\frac{u_r}{u_{max}} = 0.29 + 0.71[1 - \left(-1.5({}^x/_R)^{0.7}\right)]$	Elastic
Panet (1993, 1995)	$\frac{u_r}{u_{max}} = 0.25 + 0.75[1 - \left(\frac{0.75}{0.25 + {}^x/_R}\right)^2]$	Elastic
Chern et al. (1998)	$\frac{u_r}{u_{max}} = [1 + exp\left(\frac{-{}^x/_R}{1.1}\right)^{-1.7}]$	Elasto-plastic
Unlu and Gercek (2003)	$\frac{u_r}{u_{max}} = \frac{u_o}{u_{max}} + A_a(1 - e^{B_a({}^x/_R)}),\quad {}^x/_R \leq 0$ $\frac{u_r}{u_{max}} = \frac{u_o}{u_{max}} + A_b[1 - ((B_b + ({}^x/_R))^2],\quad {}^x/_R \geq 0$ $\frac{u_o}{u_{max}} = 0.22v + 0.19,\quad {}^x/_R = 0$ $A_a = -0.22v + 0.19 \quad B_a = 0.73v + 0.81$ $A_b = -0.22v + 0.81 \quad B_b = 0.39v + 0.65$	Elastic
Vlachopoulos and Diederichs (2009)	$\frac{u_r}{u_{max}} = \frac{u_o}{u_{max}}\ e^{{}^x/_R},\quad {}^x/_R \leq 0$ $\frac{u_r}{u_{max}} = 1 - (1 - \frac{u_o}{u_{max}})e^{(-3x/_R)/(2r_p/_R)},\quad {}^x/_R \geq 0$ $\frac{u_o}{u_{max}} = \frac{1}{3}e^{-0.15({}^{r_p}/_R)},\quad {}^x/_R = 0$ r_p – plastic radius	Elasto-plastic

Empirical relationships obtained from the results of 3D finite element analyses with a wide range of ground parameters and tunnel geometries (size and depth)

Evolution of wall convergence along the tunnel axis (x) (Chern, 1998)



$$\frac{u_R(x)}{u_{R\infty}} = \left[1 + exp\left(0.91\frac{x}{R}\right)\right]^{-1.7}$$

Chern (1998): Empirical relationship from the results of a set of 3D finite element analyses with a wide range of ground parameters and tunnel geometries (size and depth)

Evolution of wall convergence along the tunnel axis (x) (Chern, 1998)

Wall convergence $u_R(x)$ of an unsupported tunnel at distance (x) from the tunnel face (located at x = 0) :

$$\frac{u_R(x)}{u_{R\infty}} = \left[1 + exp\left(0.91 \frac{x}{R} \right) \right]^{-1.7} \quad \text{or} \quad \frac{x}{R} = 1.10 \ \ln\left[\left(\frac{u_R(x)}{u_{R\infty}} \right)^{-0.588} - 1 \right]$$

R = tunnel radius

$u_{R\infty}$ = the final (maximum) wall convergence at large distance from the tunnel face (x = -∞). Can be calculated with analytical methods (present section), but more accurately with finite element analyses

$u_R(o)$ = wall convergence at the tunnel face (location x = 0)

According to Chern: $u_R(o) = 0.308\ u_{R\infty}$

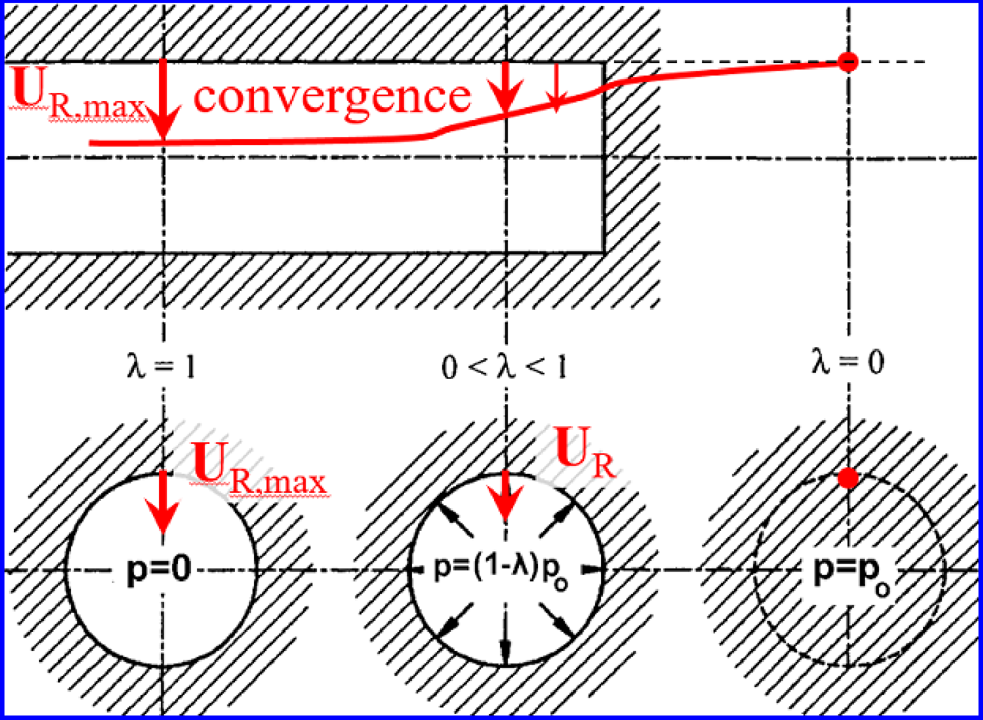
Stresses and deformations around a cylindrical tunnel - Assumptions

- 2D model of tunnel excavation: The initial geostatic pressure (p_o) gradually reduces to (p) and eventually becomes zero. As the stress reduces, the tunnel wall converges (U_R) up to a maximum value $U_{R,max}$ (when p=0).

Deconfinement = Reduction of pressure p

Deconfinement coefficient: $$\lambda = 1 - \frac{p}{p_o} \Rightarrow p = (1-\lambda)\, p_o$$

$$\lambda = 0 \rightarrow p = p_o \quad , \quad \lambda = 1 \rightarrow p = 0$$



Wall convergence U_R reaches a maximum value $U_{R,max}$ and does not continue to increase more. Why ? Because the stress change (p in 2D models) that causes ground deformation, only occurs close to the tunnel face, i.e., along the length between λ=0 and 1.

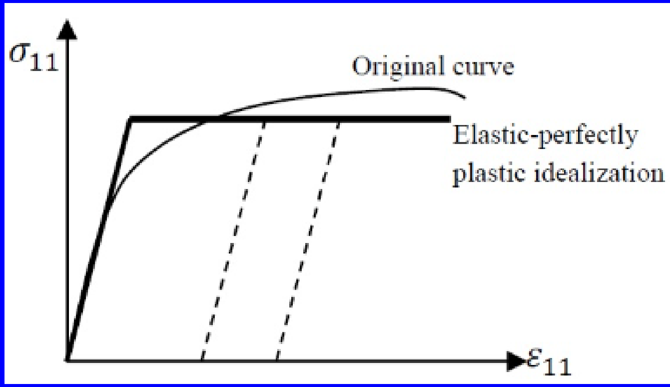
NOTE: Need a relation between p (i.e., λ) and location (x) to link 2D with 3D models

Relation U_R and p (or λ): from 2D analysis (next)
Relation U_R and x (from Chern)

} Relation p (or λ) and x

Stresses and deformations around a cylindrical tunnel - Assumptions

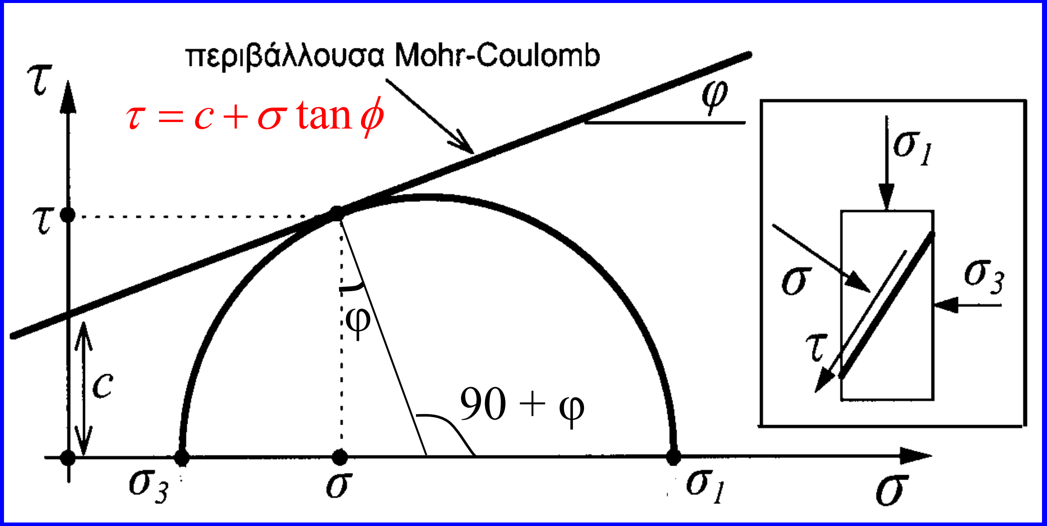
- 2D (plane) strain (no change along tunnel axis z)
- Cylindrical unsupported tunnel, with radius R
- Hydrostatic initial (geostatic) stress state ($K_o = 1 \rightarrow \sigma_{vo} = \sigma_{ho} = p_o$)
- Elastic – perfectly plastic ground, yielding with Mohr-Coulomb criterion (strength parameters: c, φ)
- Constant dilatancy (δ) in the plastic domain: $\tan\delta \equiv \dfrac{\varepsilon_r + \varepsilon_\theta}{\varepsilon_r - \varepsilon_\theta} = \dfrac{volumetric\ strain}{shear\ strain}$



Mohr-Coulomb criterion:

$$\tau = c + \sigma\tan\phi \quad \Rightarrow \quad \sigma_1 = \sigma_3 N_\varphi + 2c\sqrt{N_\varphi}$$

$$N_\phi \equiv k = \frac{1+\sin\varphi}{1-\sin\varphi} = \tan^2\left(45 + \frac{\phi}{2}\right) = \left(\frac{\cos\phi}{1-\sin\phi}\right)^2$$



Strength σ_{cm}: σ_1 for σ_3=0

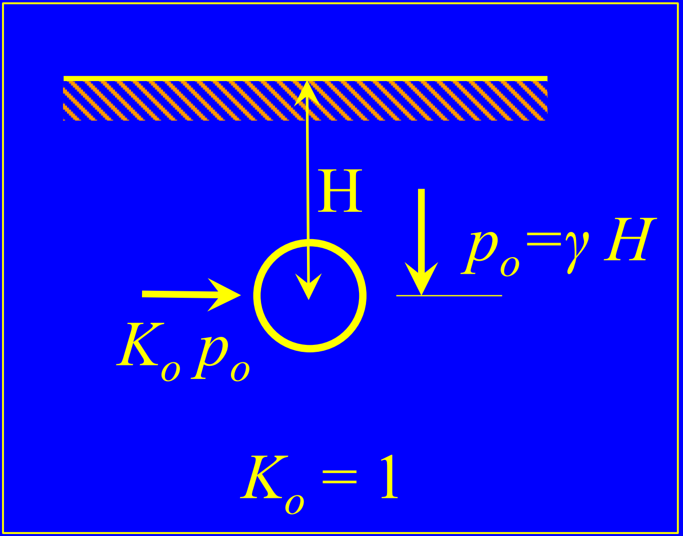
$$\sigma_{cm} = 2c\sqrt{k} = \frac{2c\cos\phi}{1-\sin\phi}$$

$$\boxed{\sigma_1 = k\sigma_3 + \sigma_{cm}}$$

Stresses and deformations around a cylindrical tunnel - Assumptions

Definitions:

Overstress factor: $N_s = \dfrac{2\,p_o}{\sigma_{cm}}$ (for K_o = 1)



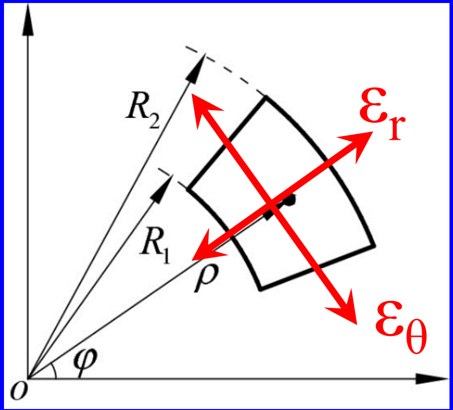
Ground strength: $\sigma_{cm} = 2c\sqrt{k} = \dfrac{2\,c\,\cos\phi}{1-\sin\phi}$

Rockmass strength (empirical correlation with GSI):

$$\sigma_{cm} = \frac{\sigma_{ci}}{50}\exp\left(\frac{GSI}{25.5}\right)$$

σ_{ci} = strength of intact rock

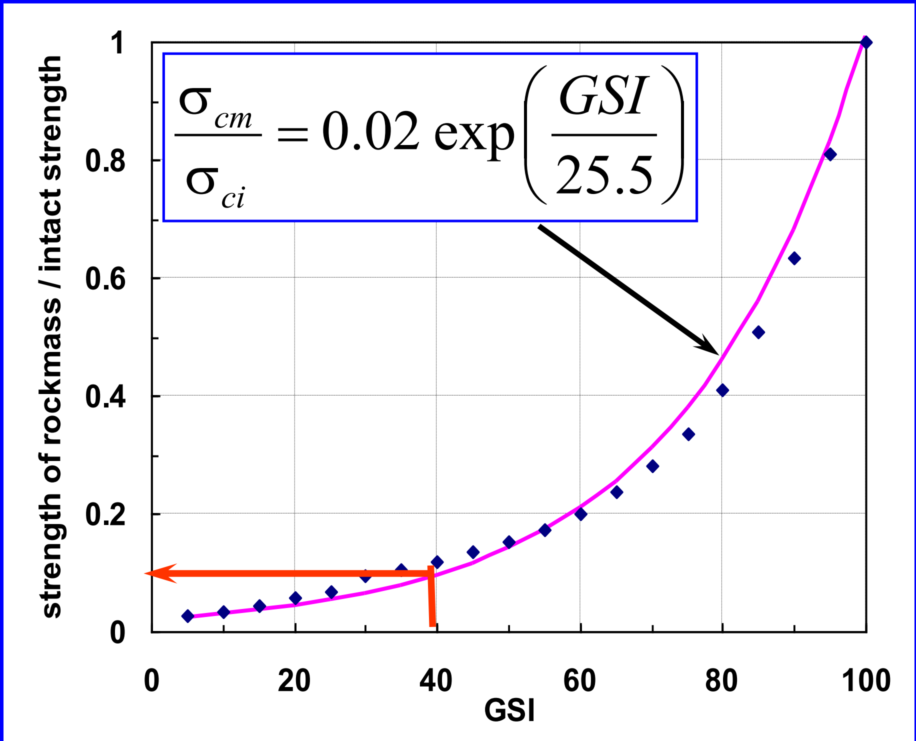
Dilatancy (δ): $\tan\delta \equiv \dfrac{\varepsilon_r + \varepsilon_\theta}{\varepsilon_r - \varepsilon_\theta}$



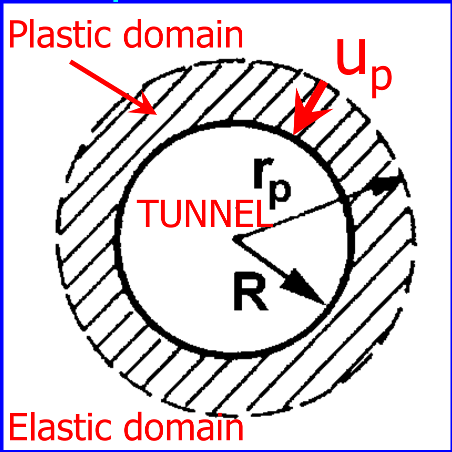
$$K \equiv \frac{1+\tan\delta}{1-\tan\delta}$$

δ=0 → K=1

δ=11° → K=1.5



Stresses and deformations around a cylindrical tunnel - Assumptions



Ground deformations in plastic domain ($r < r_p$):

Dilatancy is constant in plastic domain (parameter K):

$$\tan\delta \equiv \frac{\varepsilon_r + \varepsilon_\theta}{\varepsilon_r - \varepsilon_\theta} \quad \text{where:} \quad K \equiv \frac{1+\tan\delta}{1-\tan\delta}$$

Strain definitions (u = radial displacement):

$$\varepsilon_r = \frac{du}{dr} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = 0$$

The above formulae give (inside the plastic domain (i.e., for $r < r_p$):

$$\frac{du}{dr} + \frac{K}{r}u = 0 \Rightarrow u = c\,r^{-K} \Rightarrow c = u_p\,r_p^K \Rightarrow u = u_p\left(\frac{r_p}{r}\right)^K \quad \text{For } R < r < r_p$$

(at $r = r_p \rightarrow u = u_p$)

u_p is the radial displacement at $r = r_p$ (calculated from the elastic zone)

At the tunnel wall (r = R) : $u_R = u_p\left(\dfrac{r_p}{R}\right)^K$

Stresses and deformations around a cylindrical tunnel – Elastic domain ($r > r_p$)

Stress-strain relationships in plane strain (cylindrical coordinates):

$$\varepsilon_r = \frac{1}{\Lambda}\{ \text{[illegible]}$$

$$\varepsilon_\theta = \frac{1}{\Lambda}\{\dot{\sigma}_\theta - K_o \dot{\sigma}_r\}$$

$$\Lambda \equiv \frac{E}{(1+\nu)(1-\nu)} \qquad K_o \equiv \frac{\nu}{1-\nu}$$

$$\dot{\sigma}_r = \sigma_r - p_o \qquad \dot{\sigma}_\theta = \sigma_\theta - p_o$$

Solving for the stress increments:

$$\dot{\sigma}_r = D\{\varepsilon_r + K_o \varepsilon_\theta\} \qquad \dot{\sigma}_\theta = D\{\varepsilon_\theta + K_o \varepsilon_r\} \qquad D \equiv \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

Equilibrium equation (along axis r):

$$\frac{d\dot{\sigma}_r}{dr} + \frac{\dot{\sigma}_r - \dot{\sigma}_\theta}{r} = 0$$

Strain definitions (u = radial displacement):

$$\varepsilon_r = \frac{du}{dr} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = 0$$

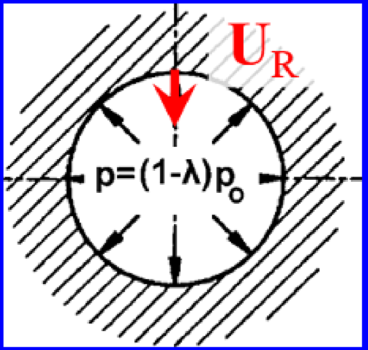
$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \quad \Rightarrow \quad u = c_1 r + \frac{c_2}{r}$$

Boundary conditions: $c_1 = 0$ (u cannot increase with r)

If plastic zone exists: At $r = r_p$ → $u = u_p$ → $c_2 = u_p r_p$ → $u = u_p\left(\frac{r_p}{r}\right)$ (in elastic zone)

If plastic zone does not exist: At r=R → $\sigma_r = p$ → $c_2 = \lambda \frac{p_o R^2}{2G}$ → $u = \lambda R\left(\frac{p_o}{2G}\right)\frac{R}{r}$

Deformations around a cylindrical tunnel



1. Linearly elastic ground, $K_o = 1$

The differential equation of equilibrium gives: $u = \dfrac{c_2}{r}$

Constant c_2 is determined from the stress boundary condition:

$$\sigma_r(r = R) = p = p_o(1 - \lambda)$$

Thus, the radial displacement at distance (r) is:

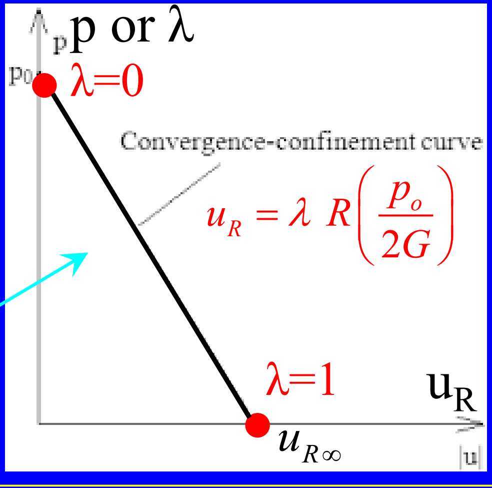
$$u = \lambda\ R\left(\frac{p_o}{2G}\right)\left(\frac{R}{r}\right) \Rightarrow u = \left(1 - \frac{p}{p_o}\right)R\left(\frac{p_o}{2G}\right)\left(\frac{R}{r}\right)$$

NOTE: Strains and stresses calculated by differentiation

At the tunnel wall $(r=R)$: $u_R = \lambda\ R\left(\dfrac{p_o}{2G}\right)$

and for complete deconfinement ($\lambda=1$, $r=R$) : $u_{R\infty} = R\left(\dfrac{p_o}{2G}\right)$

$$\frac{u_R}{u_{R\infty}} = \lambda$$



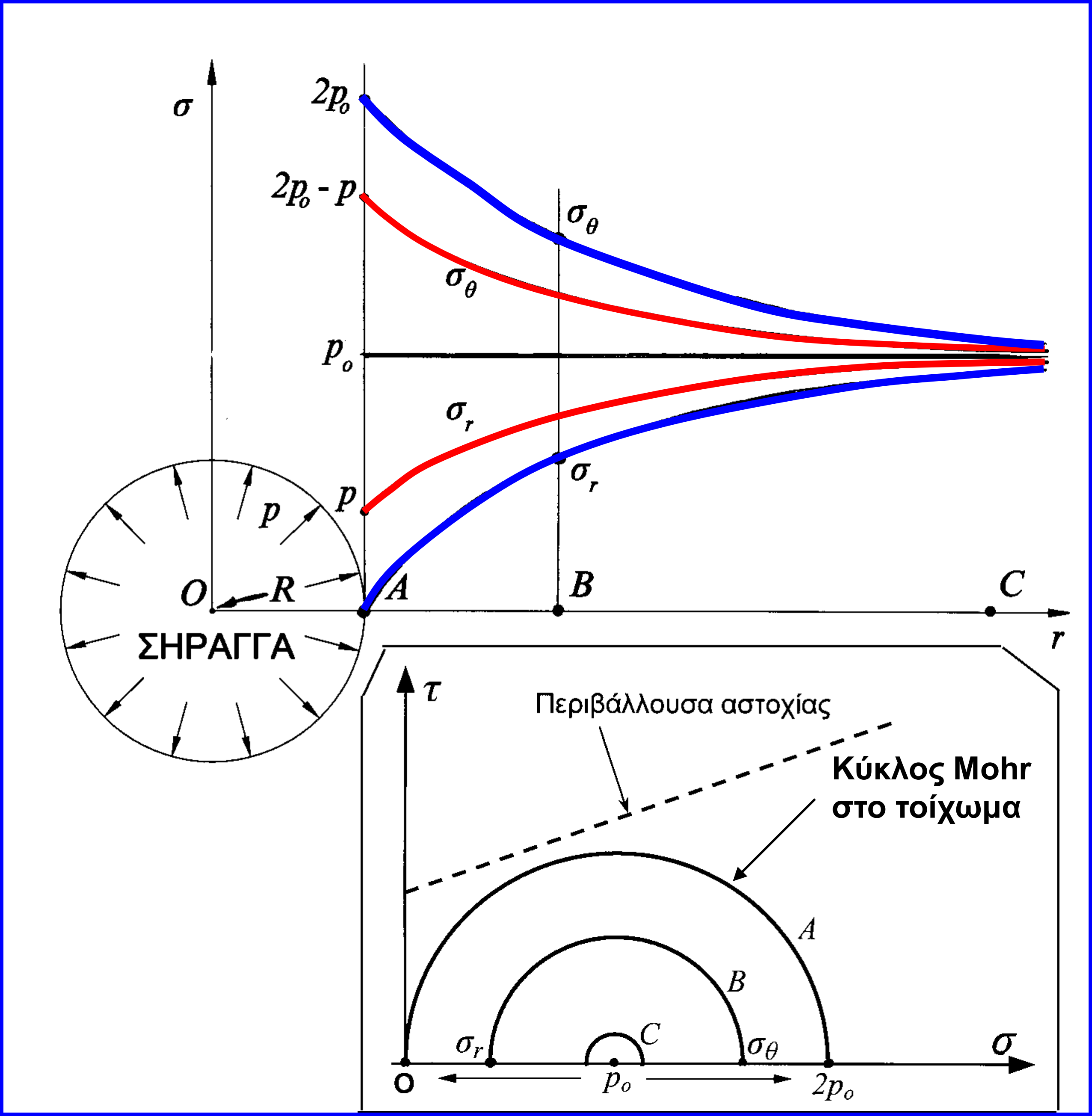
Convergence-confinement curve in linearly elastic ground

G = ground shear modulus $\qquad G = \dfrac{E}{2(1+\nu)}$ $\qquad \lambda = 1 - \dfrac{p}{p_o}$

R = tunnel radius , $\quad p_o$ = geostatic stress

$\lambda = \lambda\ (x)$ = deconfinement coefficient $\qquad 0 < \lambda < 1$

Stresses and deformations around a cylindrical tunnel – Only elasticity



Linearly elastic ground

$K_o = 1$

$$\sigma_r = p_o\left[1-\lambda\left(\frac{R}{r}\right)^2\right]$$

$$\sigma_\theta = p_o\left[1+\lambda\left(\frac{R}{r}\right)^2\right]$$

$$\lambda = 1-\frac{p}{p_o}$$

At tunnel wall (r=R):

$$\sigma_r = p = (1-\lambda)p_o$$

$$\sigma_\theta = 2p_o - p = (1+\lambda)p_o$$

and for λ=1:

$\sigma_r = 0\ ,\ \ \sigma_\theta = 2p_o$

Stresses and deformations around a cylindrical tunnel – Only elasticity

Linearly elastic ground – $K_o \neq 1$ (Kirsch solution)

Circular tunnel (radius r_o) at depth (H), unit weight of ground (γ), horizontal stress coefficient K ($\sigma_h = K\,\sigma_v$). Geostatic stresses: $\sigma_v = \gamma H$, $\sigma_h = K\gamma H$ (do not vary with depth).

Angle (θ) is measured from tunnel center, with respect to the vertical (θ=0)

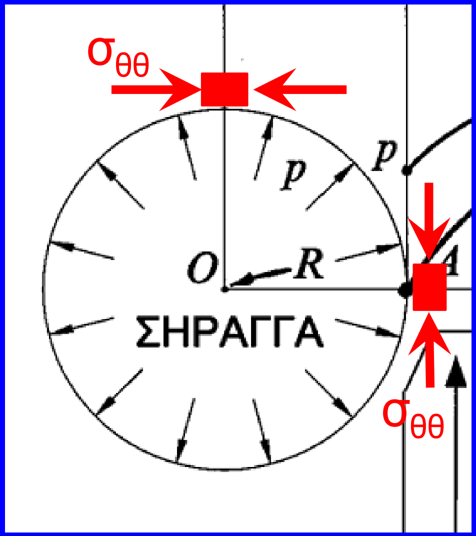
Tunnel is unsupported and λ = 1 ($\sigma_{rr} = 0$ at $r = r_o$)

Kirsch solution (for p=0):

$$\sigma_{rr} = \gamma H\left[\frac{1+K}{2}\left(1-\frac{r_0^2}{r^2}\right)\right] + \gamma H\left[\frac{1-K}{2}\left(1+3\frac{r_0^4}{r^4}-4\frac{r_0^2}{r^2}\right)\cos 2\vartheta\right]$$

$$\sigma_{\vartheta\vartheta} = \gamma H\left[\frac{1+K}{2}\left(1+\frac{r_0^2}{r^2}\right)\right] - \gamma H\left[\frac{1-K}{2}\left(1+3\frac{r_0^4}{r^4}\right)\cos 2\vartheta\right]$$

$$\sigma_{r\vartheta} = -\gamma H\frac{1-K}{2}\left(1-3\frac{r_0^4}{r^4}+2\frac{r_0^2}{r^2}\right)\sin 2\vartheta$$



Circumferential stress at springline (θ=90°): $\sigma_{\theta\theta} = (3-K)\gamma H$ - Initial value: $\sigma_{\theta\theta} = \gamma H$

For K = 0.5 -> $\boldsymbol{\sigma_{\theta\theta} = 2.5\,\gamma H}$

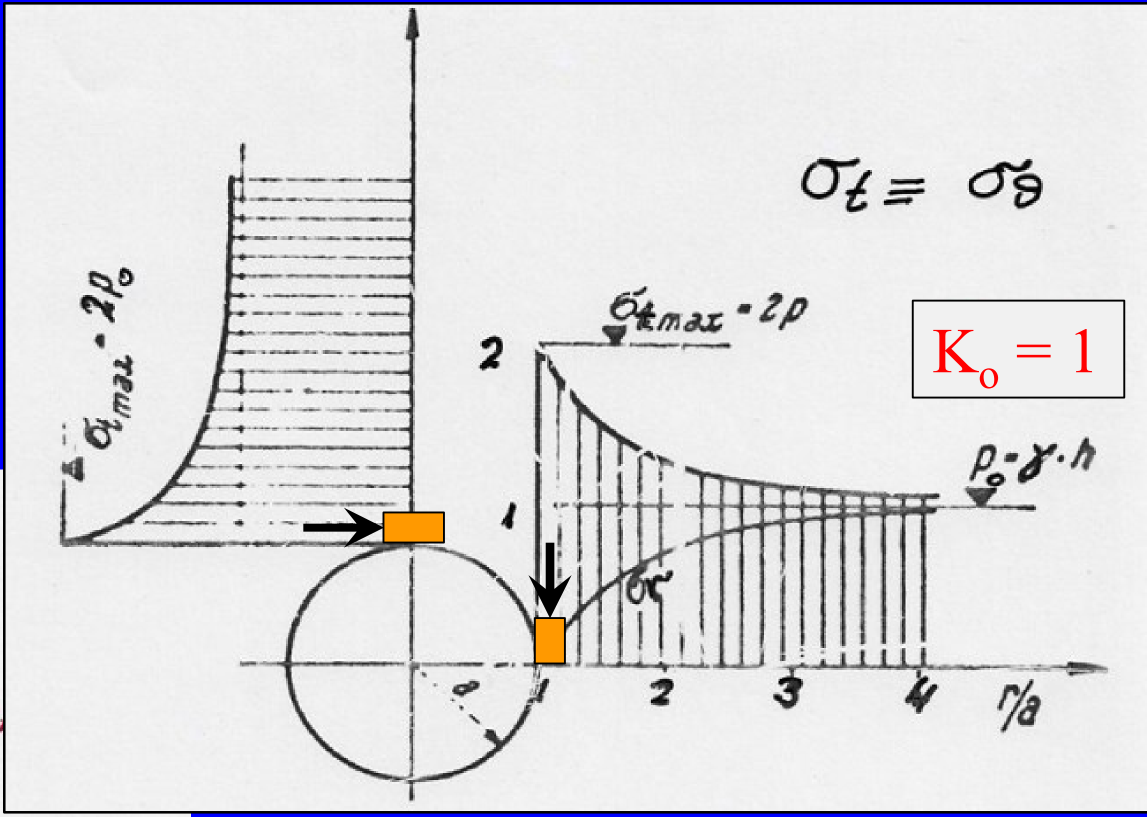
K = 1 -> $\sigma_{\theta\theta} = 2\,\gamma H$

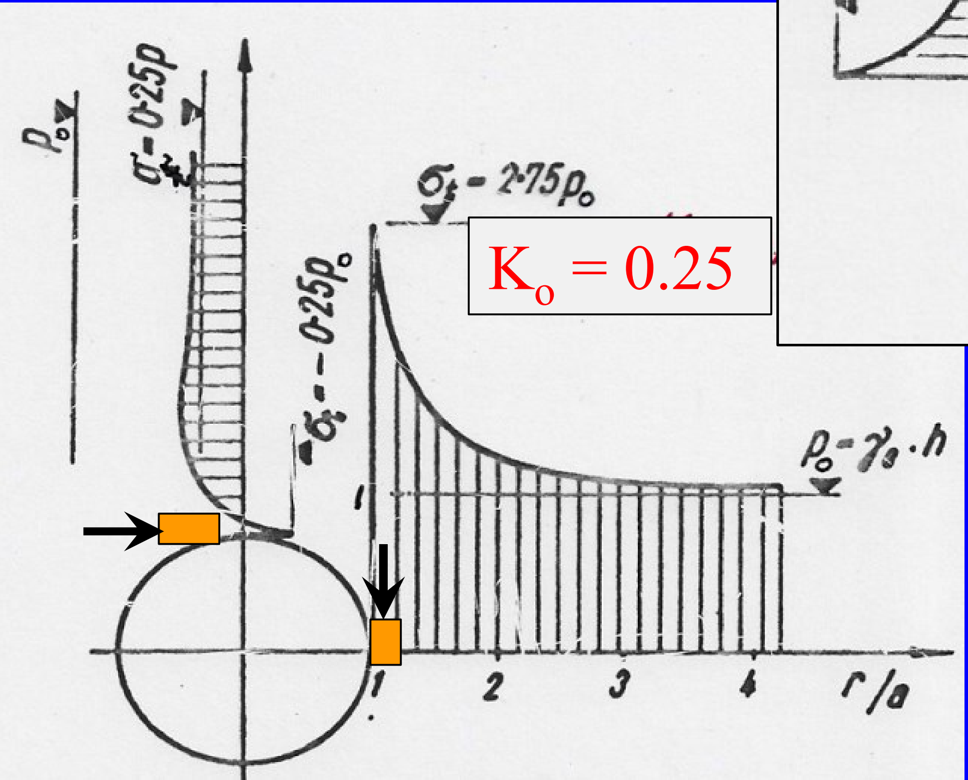
Circumferential stress at crest and invert (θ=0 & 180°): $\sigma_{\theta\theta} = (3K-1)\gamma H$ - Initial value: $\sigma_{\theta\theta} = K\gamma H$

For K = 0.5 -> $\boldsymbol{\sigma_{\theta\theta} = 0.5\,\gamma H}$ (initial value 0.5 γH)

K = 1 -> $\sigma_{\theta\theta} = 2\,\gamma H$ (initial value γH)

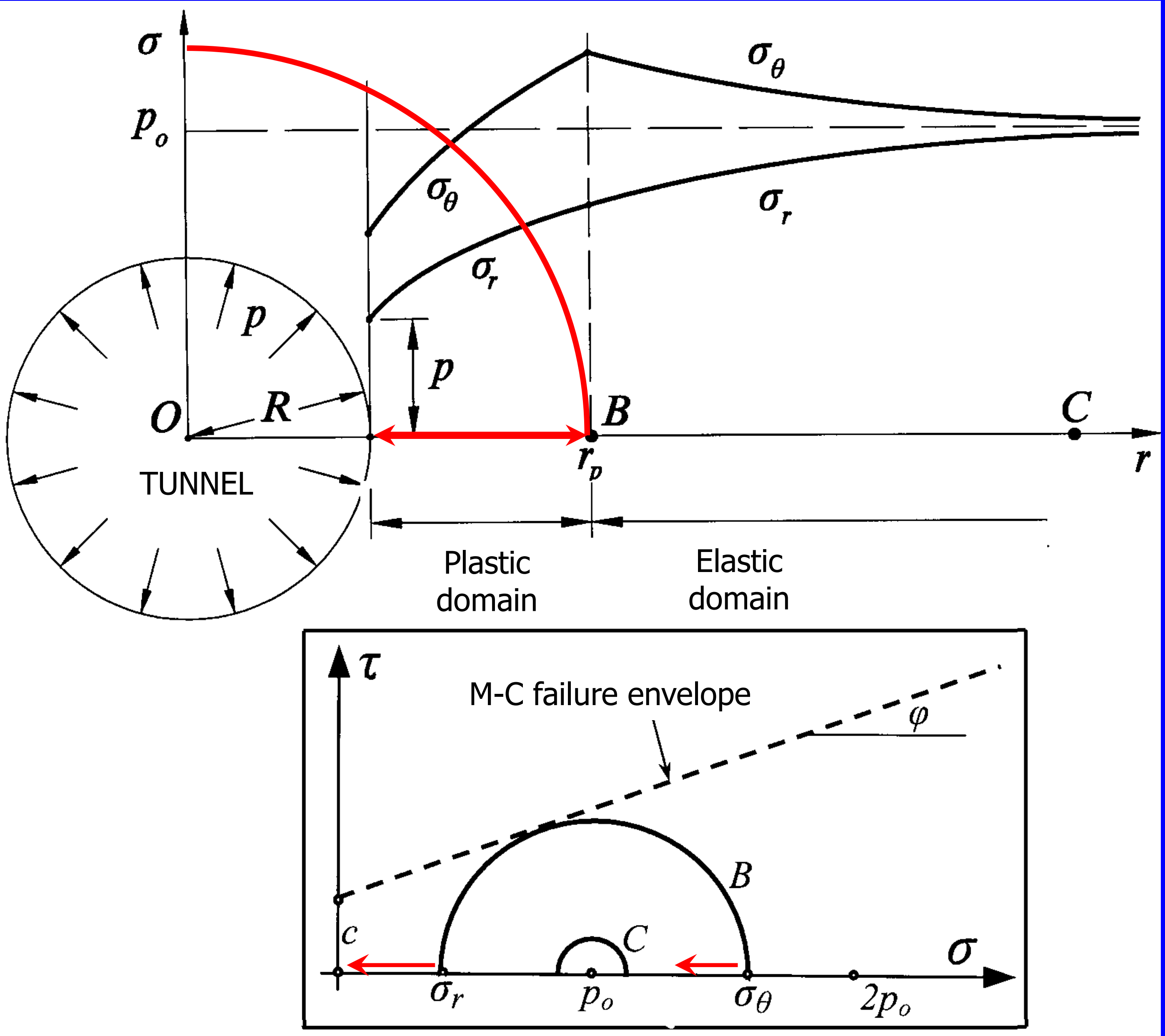
Stresses and deformations around a cylindrical tunnel – Only elasticity
Linearly elastic ground – $K_o \neq 1$ (Kirsch solution)





Stresses and deformations around a cylindrical tunnel

2. Elastic – perfectly plastic ground, $K_o = 1$

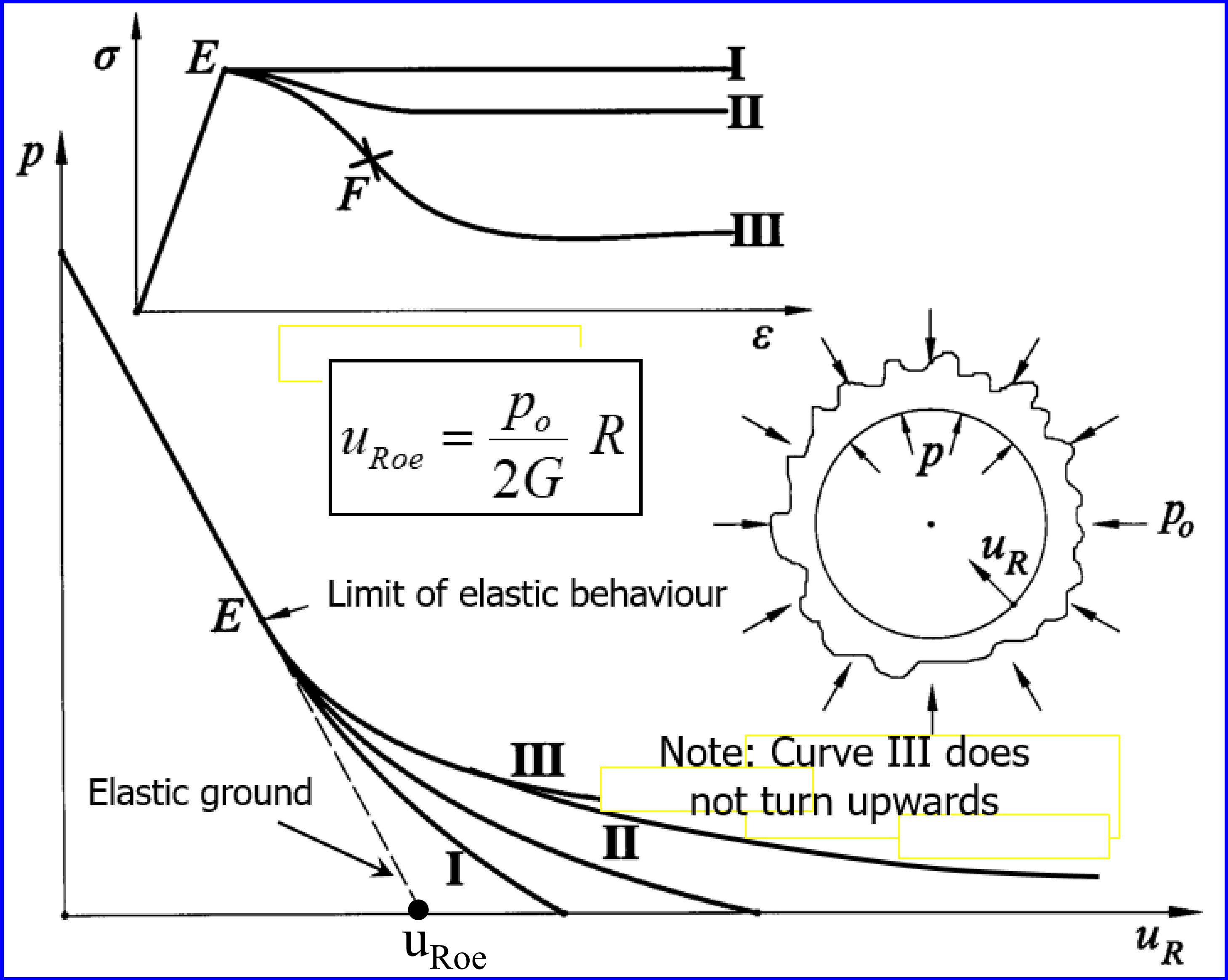


The limit of the plastic zone (r_p) depends on:

- The tunnel radius (R)
- the ground strength parameters (c,φ)
- the initial geostatic stress (p_o)
- the deconfinement coefficient (λ), i.e., the internal pressure (p)

Convergence – confinement curve in elasto-plastic ground

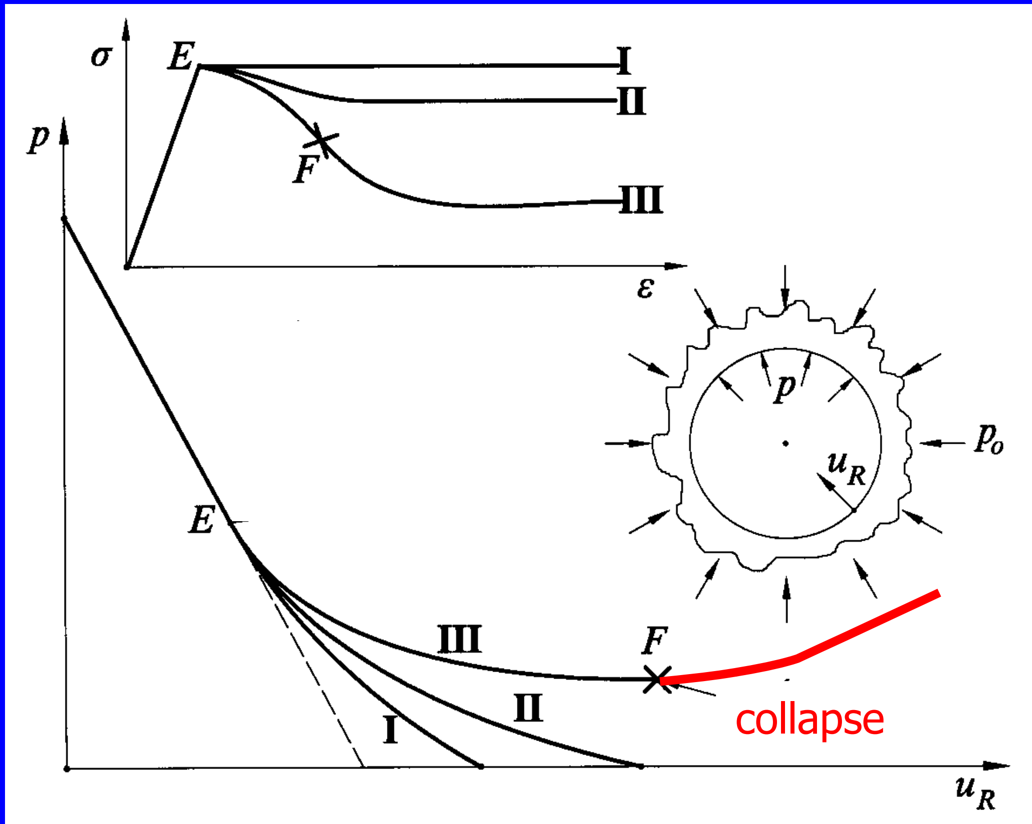
Influence of the σ-ε curve

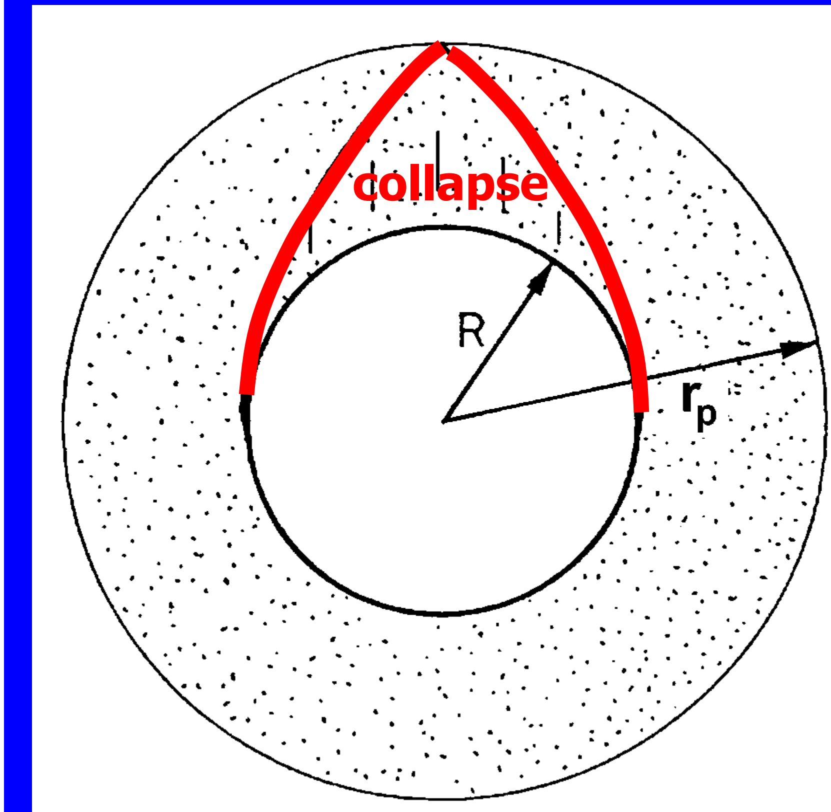


The ground pressure (p) on the tunnel lining decreases with increasing tunnel wall convergence

Convergence – confinement curve in elasto-plastic ground

Influence of the σ-ε curve



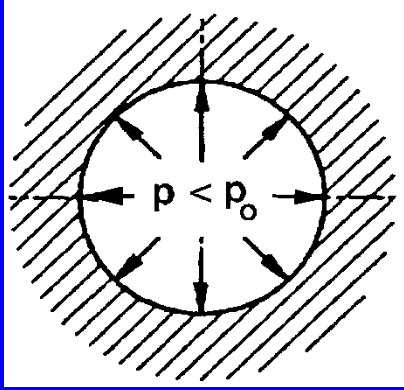


If ground continuity is preserved, the convergence-confinement curve does NOT turn upward (collapse) even in strongly strain softening ground. If, however, ground continuity is lost (e.g. rock block contact is lost) due to large ground deformations, then the convergence-confinement curve may turn upwards (collapse).

This means that ground pressure on the tunnel lining will increase at large ground deformations.

Stresses around a cylindrical tunnel

2. Elastic – perfectly plastic ground, $K_o = 1$



Calculation of the minimum internal pressure $p = p_{cr}$ to maintain elasticity in the ground:

Stress distribution in the elastic domain:

$$\sigma_r = p_o\left[1-\lambda\left(\frac{R}{r}\right)^2\right] \qquad \sigma_\theta = p_o\left[1+\lambda\left(\frac{R}{r}\right)^2\right] \qquad \lambda = 1-\frac{p_{cr}}{p_o}$$

Stresses (elastic) at the tunnel wall (r=R):

$$\sigma_1 = \sigma_\theta = 2p_o - p_{cr}$$
$$\sigma_3 = \sigma_r = p_{cr}$$

Marginal fulfillment of the M-C failure criterion at the tunnel wall:

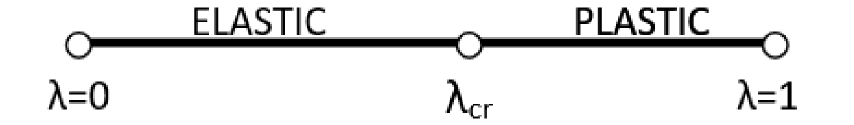
$$\sigma_1 = k\,\sigma_3 + \sigma_{cm} \quad \Rightarrow \quad \frac{p_{cr}}{p_o} = \left(\frac{2}{1+k}\right)\left(\frac{N_s-1}{N_s}\right)$$

Critical deconfinement coefficient: $\lambda_{cr} = 1-\frac{p_{cr}}{p_o} = 1-\left(\frac{2}{1+k}\right)\left(\frac{N_s-1}{N_s}\right)$

CONCLUSION: There is no plastic zone around the tunnel, If: $\lambda_{cr} \geq 1$ (i.e., $N_s \leq 1$) or

if: $\lambda_{cr} < 1$ and $\lambda \leq \lambda_{cr}$

Plastic zone develops around the tunnel if: $\lambda_{cr} < 1$ (i.e., $N_s > 1$) and $\lambda > \lambda_{cr}$



Stresses around a cylindrical tunnel – elastoplastic ground

Critical deconfinement coefficient – ground remains elastic but M-C failure criterion is marginally fulfilled at the tunnel wall (i.e., $r_p=R$):

$$\lambda_{cr} = 1 - \frac{p_{cr}}{p_o} \Rightarrow \lambda_{cr} = 1 - \left(\frac{2}{1+k}\right)\left(\frac{N_s - 1}{N_s}\right)$$

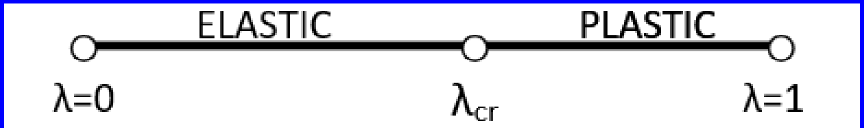
$$N_s = 2p_o / \sigma_{cm}$$

$$k = \tan^2\left(45 + \frac{\varphi}{2}\right)$$

$$\sigma_{cm} = \frac{\sigma_{ci}}{52.63}\exp\left(\frac{GSI}{20}\right)$$

Values of λ_{cr} (plastic zone around the tunnel develops if $\lambda > \lambda_{cr}$)

φ (deg)	N_s = 1	N_s = 2.5	N_s = 5	N_s = 10	N_s = 15	N_s = 20
20	1.0	0.61	0.47	0.41	0.41	0.39
25	1.0	0.65	0.54	0.48	0.48	0.46
30	1.0	0.70	0.60	0.55	0.55	0.53
35	1.0	0.74	0.66	0.62	0.62	0.60
40	1.0	0.79	0.71	0.68	0.68	0.67



Stresses around a cylindrical tunnel – elastoplastic ground

Example:

γ = 22 kN/m^3, H = 100 m, K_o = 0.60 $\Rightarrow$ p_o = 0.5 (1+K_o) γ H = 1.76 MPa

GSI = 25, σ_{ci} = 12 MPa, E_i = 13.5 GPa $\Rightarrow$ σ_{cm} = 0.64 MPa , E = 821 MPa

ν = 0.30 $\Rightarrow$ G = 316 MPa

φ = 32^o $\Rightarrow$ k = 3.2546

δ = 7^o $\Rightarrow$ K = 1.28

$$G = \frac{E}{2(1+\nu)}$$

$$\sigma_{cm} = \frac{\sigma_{ci}}{52.63} \exp\left(\frac{GSI}{20}\right)$$

$$E_{\mathrm{rm}} = E_{\mathrm{i}}\left(0.02 + \frac{1 - D/2}{1 + e^{((60+15D-\mathrm{GSI})/11)}}\right)$$

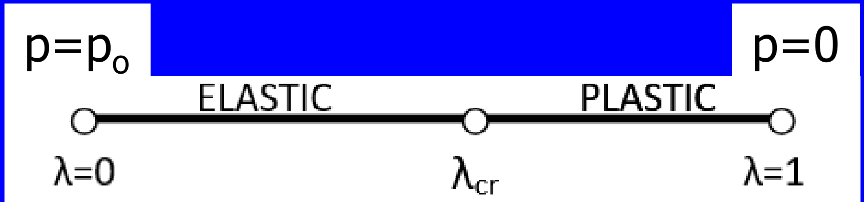
D = damage factor (=0)

$$k = \tan^2\left(45 + \frac{\varphi}{2}\right) \qquad K \equiv \frac{1+\tan\delta}{1-\tan\delta}$$

Calculations:

$$N_s = \frac{2\,p_o}{\sigma_{cm}} = 5.5 \qquad \lambda_{cr} = 1 - \left(\frac{2}{1+k}\right)\left(\frac{N_s - 1}{N_s}\right) = 0.615$$

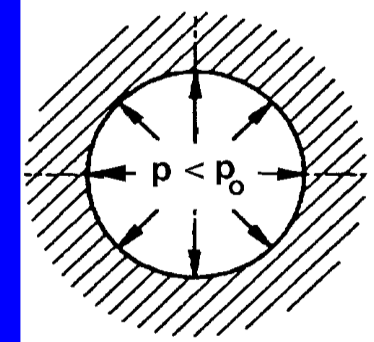
Result: For λ>0.615, i.e., for p/p_o < 0.385 plastic zone develops around the tunnel



CASE 1: Ground remains elastic (no plastic zone)

- If $N_s \leq 1 \rightarrow$ for all λ
- If $N_s > 1 \rightarrow$ for $\lambda \leq \lambda_{cr}$

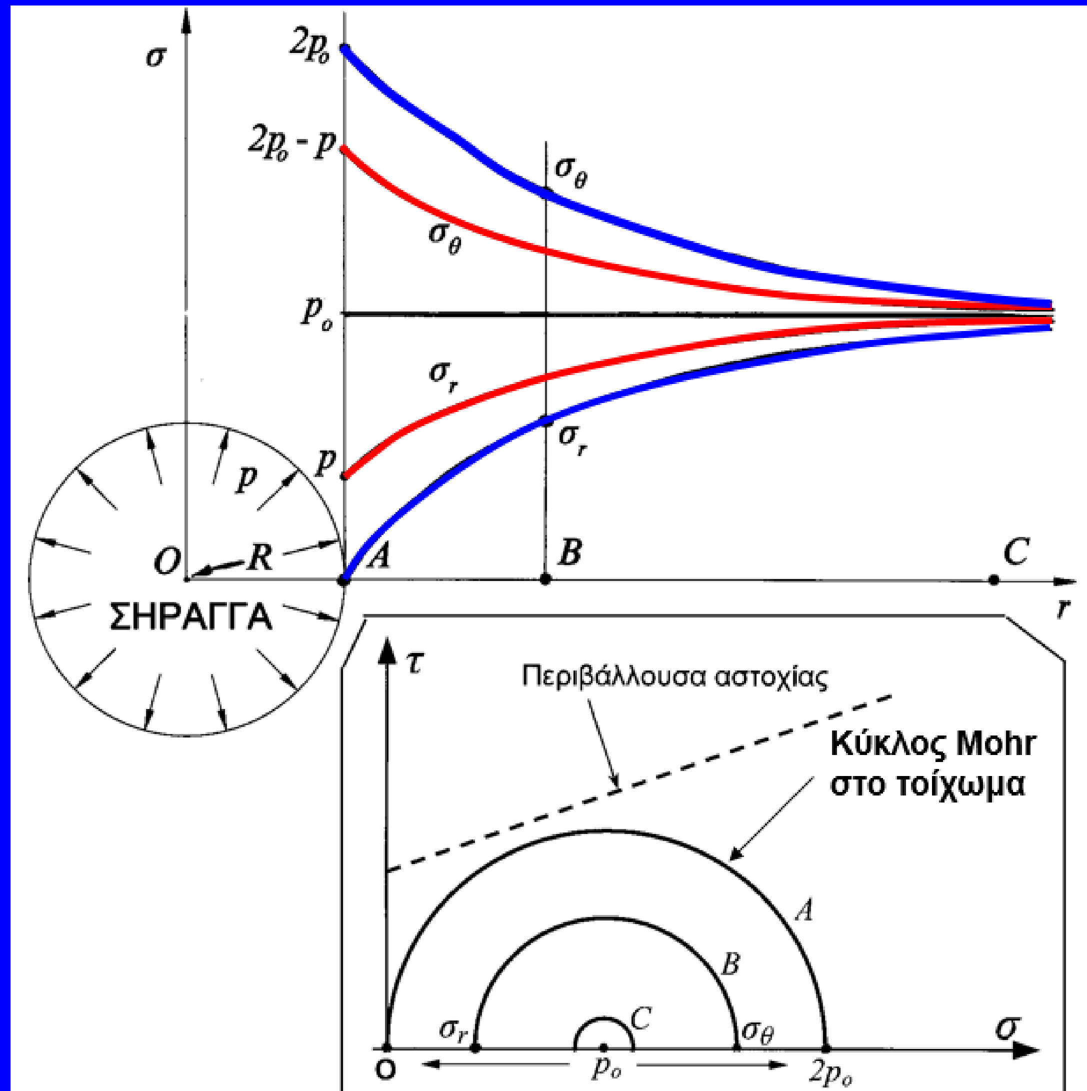
$$\lambda_{cr} = 1 - \left(\frac{2}{1+k}\right)\left(\frac{N_s - 1}{N_s}\right)$$



Stresses around the tunnel:

$$\sigma_r = p_o\left[1 - \lambda\left(\frac{R}{r}\right)^2\right]$$

$$\sigma_\theta = p_o\left[1 + \lambda\left(\frac{R}{r}\right)^2\right]$$



Displacement around the tunnel:

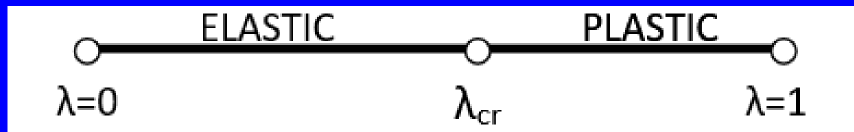
$$u = \lambda\ R\left(\frac{p_o}{2G}\right)\left(\frac{R}{r}\right)$$

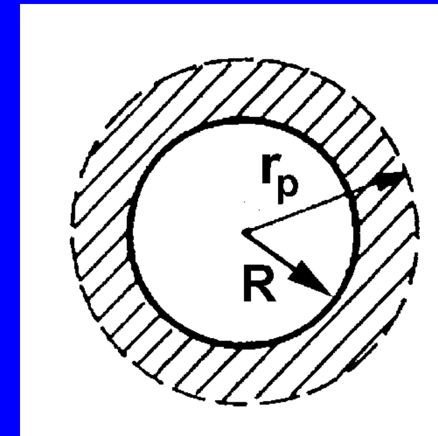
At the tunnel wall ($r=R$):

$$u_R = \lambda\ R\left(\frac{p_o}{2G}\right)$$

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$





$$\lambda = 1 - \frac{p}{p_o}$$

$$N_s = \frac{2\,p_o}{\sigma_{cm}}$$

Radius of plastic zone (r_p) :

1. If φ = 0 :

$$\frac{r_p}{R} = \exp\left[\frac{1}{2}\left(\lambda\ N_s - 1\right)\right]$$

2. If φ > 0 :

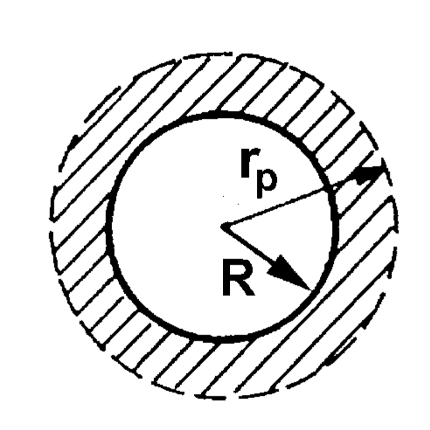
$$\frac{r_p}{R} = \left\{\left(\frac{2}{k+1}\right)\left[\frac{2 + N_s\left(k-1\right)}{2 + N_s\left(k-1\right)\left(1-\lambda\right)}\right]\right\}^{\frac{1}{k-1}}$$

And in full deconfinement (λ=1) :

$$\frac{r_{p\infty}}{R} = \left\{\left(\frac{1}{k+1}\right)\left[2 + N_s\left(k-1\right)\right]\right\}^{\frac{1}{k-1}}$$

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$



Proof of formulae for σ_r and σ_θ :

Επίλυση στην πλαστική ζώνη, δηλαδή για $R < r < r_p$

Εξίσωση ισορροπίας : $\dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0$

Κριτήριο αστοχίας Mohr-Coulomb : $\sigma_\theta = k\sigma_r + \sigma_{cm}$

Απαλειφή του σ_θ δίνει : $\dfrac{d\sigma_r}{dr} - \dfrac{1}{r}(k-1)\sigma_r - \dfrac{1}{r}\sigma_{cm} = 0$

Επίλυση της ανωτέρω διαφορικής εξίσωσης :

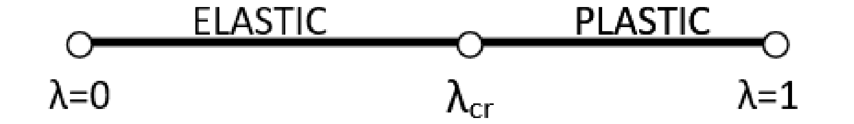
(α) **Περίπτωση :** $k \neq 1 \Rightarrow \varphi \neq 0$:

Με συνοριακή συνθήκη: $\sigma_r(r = R) = p = (1-\lambda)p_o$

$$\sigma_r = \left[(1-\lambda)p_o + \left(\frac{\sigma_{cm}}{k-1}\right)\right]\left(\frac{r}{R}\right)^{k-1} - \left(\frac{\sigma_{cm}}{k-1}\right)$$

δηλαδή : $\dfrac{\sigma_r}{p_o} = \left[(1-\lambda) + \dfrac{2}{(k-1)N_s}\right]\left(\dfrac{r}{R}\right)^{k-1} - \dfrac{2}{(k-1)N_s}$

$$\sigma_\theta = k\sigma_r + \sigma_{cm} \quad \Rightarrow \quad \frac{\sigma_\theta}{p_o} = k\frac{\sigma_r}{p_o} + \frac{2}{N_s}$$



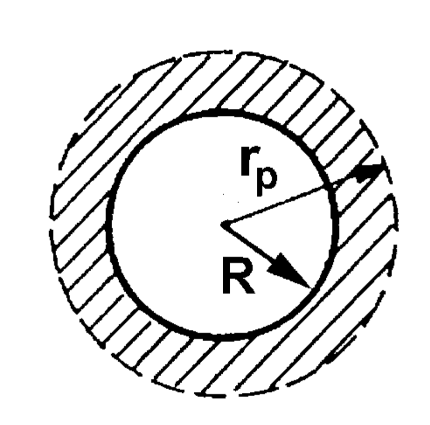
(β) **Περίπτωση :** $k = 1 \Rightarrow \varphi = 0$:

Με συνοριακή συνθήκη: $\sigma_r(r = R) = p = (1-\lambda)p_o$

$$\sigma_r = (1-\lambda)p_o + \sigma_{cm}\ln\left(\frac{r}{R}\right) \quad , \quad \sigma_\theta = \sigma_r + \sigma_{cm}$$

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$



Proof of formulae for r_p / R :

Εξίσωση των τιμών των σ_r και σ_θ στο όριο μεταξύ ελαστικής και πλαστικής ζώνης ($r = r_p$) δίνει τις τιμές των c_2 και r_p :

(α) Περίπτωση $k \neq 1 \Rightarrow \varphi \neq 0$:

$$\frac{\sigma_r}{p_o} = \left[(1-\lambda) + \frac{2}{(k-1)N_s} \right] \left(\frac{r_p}{R} \right)^{k-1} - \frac{2}{(k-1)N_s} = 1 - c_2 \left(\frac{2G}{p_o} \right) \frac{1}{r_p^{\,2}}$$

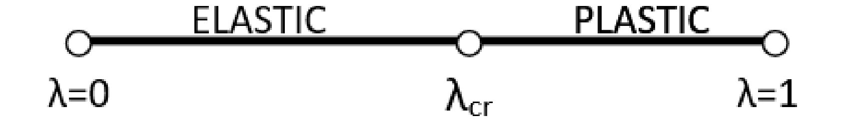
$$\frac{\sigma_\theta}{p_o} = k \frac{\sigma_r}{p_o} + \frac{2}{N_s} = 1 + c_2 \left(\frac{2G}{p_o} \right) \frac{1}{r_p^{\,2}}$$

οπότε :

$$\frac{r_p}{R} = \left[\left(\frac{2}{k+1} \right) \frac{N_s + \frac{2}{k-1}}{(1-\lambda)N_s + \frac{2}{k-1}} \right]^{\frac{1}{k-1}}$$

(β) Περίπτωση : $k = 1 \Rightarrow \varphi = 0$:

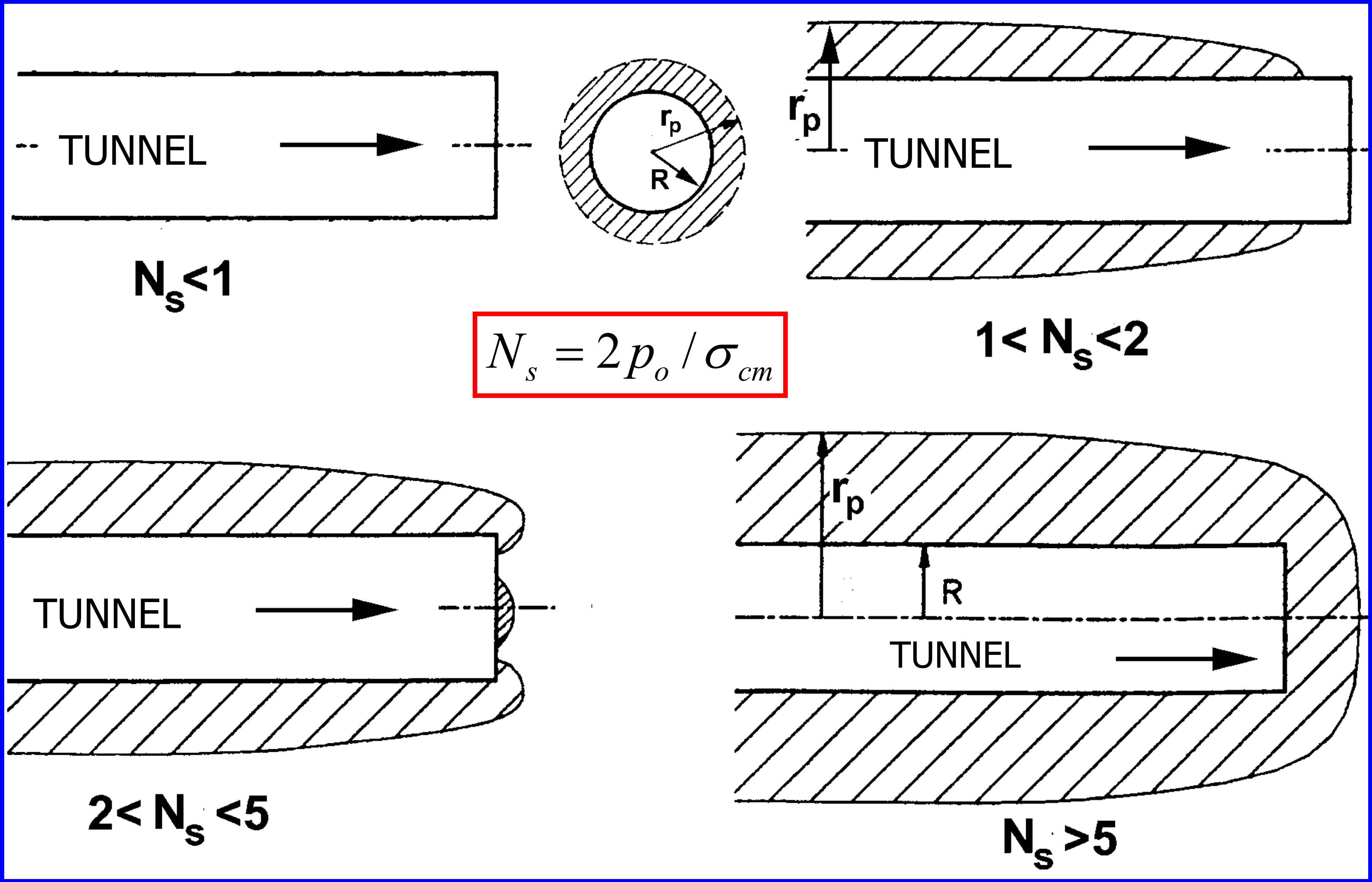
$$\frac{r_p}{R} = \exp\left[\frac{1}{2} (N_s \lambda - 1) \right]$$



Stresses around a cylindrical tunnel – elastoplastic ground

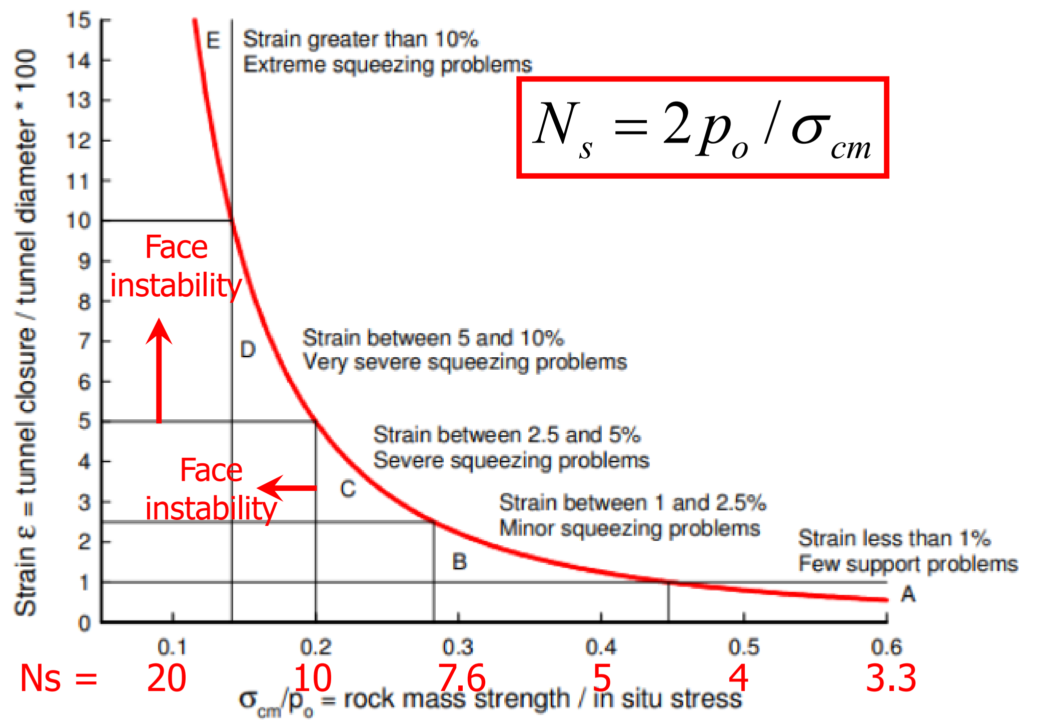
Ground remains elastic, always (for all λ) if: $N_s \leq 1$

Schematic size of the plastic zone (r_p) around the tunnel



Stresses around a cylindrical tunnel – elastoplastic ground

Classification of tunnel excavation problems with N_s value

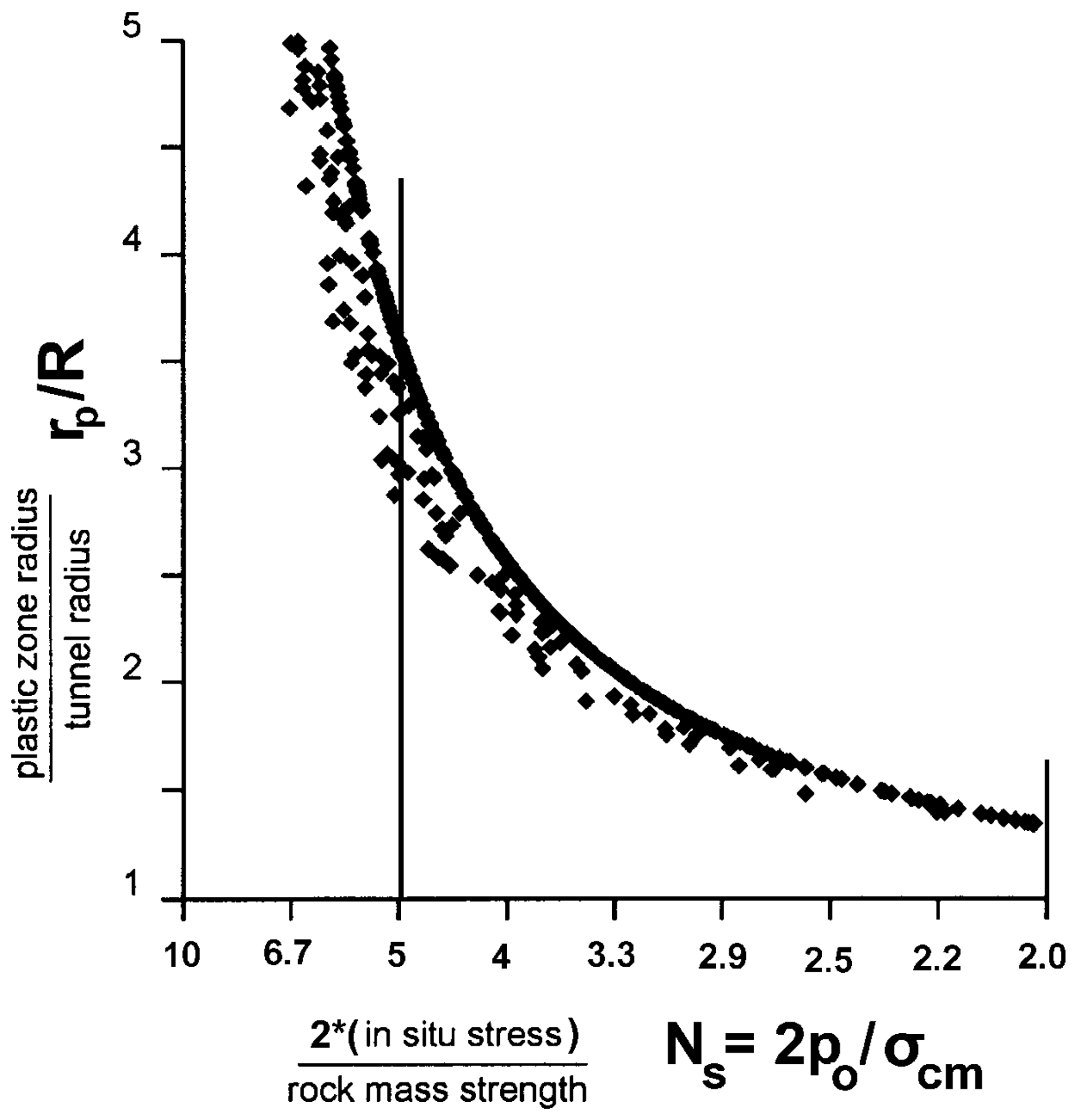


	Strain ε %	Geotechnical issues
A	Less than 1 Ns < 4.5	Few stability problems and very simple tunnel support design methods can be used. Tunnel support recommendations based upon rock mass classifications provide an adequate basis for design.
B	1 to 2.5 Ns = 4.5 to 8	Convergence confinement methods are used to predict the formation of a 'plastic' zone in the rock mass surrounding a tunnel and of the interaction between the progressive development of this zone and different types of support.
C	2.5 to 5 Ns = 8 to 10	Two-dimensional finite element analysis, incorporating support elements and excavation sequence, are normally used for this type of problem. Face stability is generally not a major problem.
D	5 to 10 Ns = 10 to 16	The design of the tunnel is dominated by face stability issues and, while two-dimensional finite analyses are generally carried out, some estimates of the effects of forepoling and face reinforcement are required.
E	More than 10 Ns > 16	Severe face instability as well as squeezing of the tunnel make this an extremely difficult three-dimensional problem for which no effective design methods are currently available. Most solutions are based on experience.

Hoek E and Marinos P 2000 Predicting Tunnel Squeezing Problems in Weak Heterogeneous Rock Masses. *Tunnels and Tunnelling International* **32(11)** 45-51

Stresses around a cylindrical tunnel – elastoplastic ground

Radius of the plastic zone r_p (unsupported tunnel)



p_o = geostatic stress

λ = deconfinement coefficient

σ_{cm} = ground strength

N_s = overstress factor

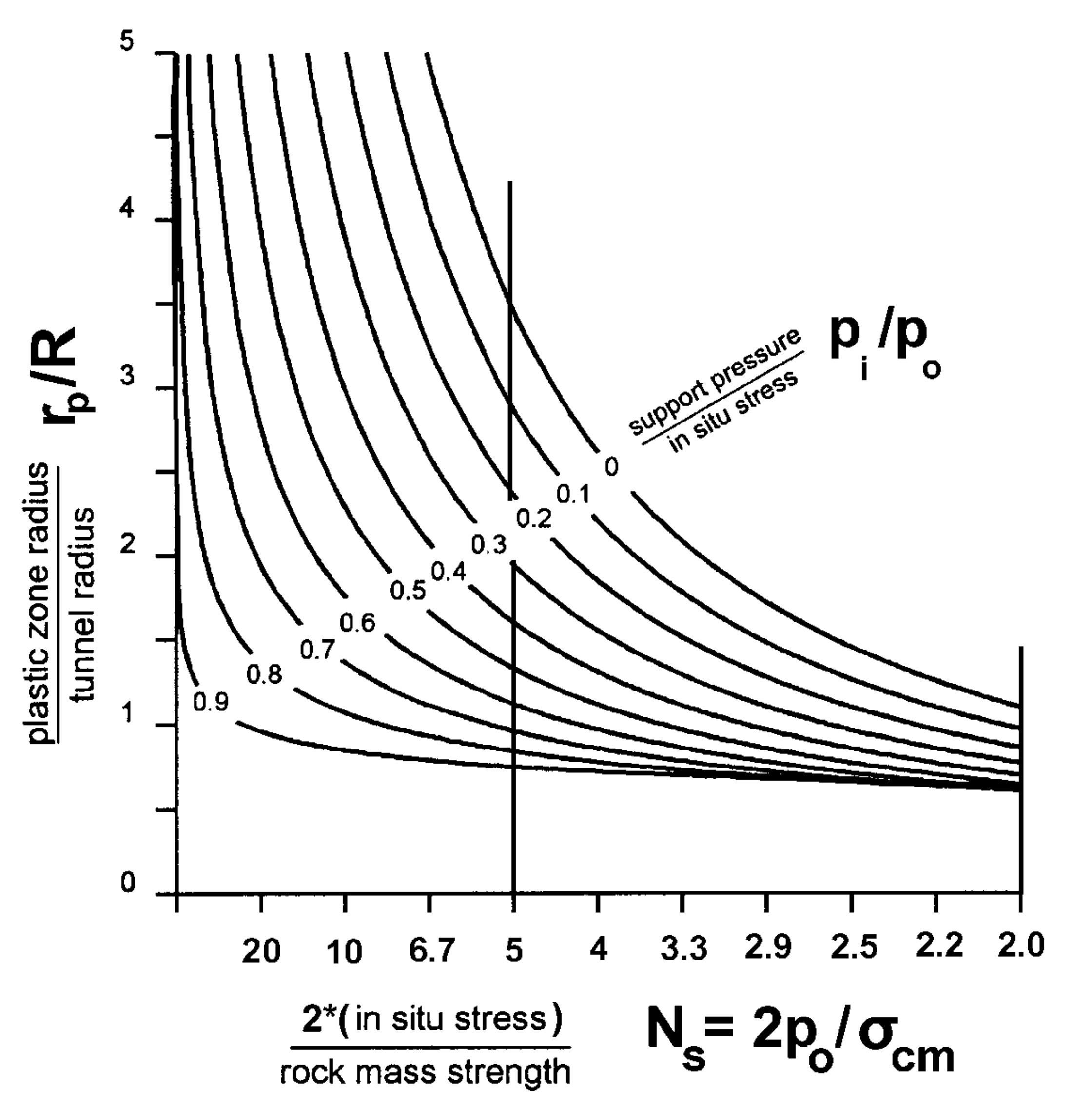
$$p = p_o(1-\lambda)$$

$$N_s = \frac{2\,p_o}{\sigma_{cm}}$$

Graph is valid for common values of the relevant parameters

Stresses around a cylindrical tunnel – elastoplastic ground

Radius of the plastic zone r_p (supported tunnel, p_i = support pressure)



p_o = geostatic stress

λ = deconfinement coefficient

σ_{cm} = ground strength

N_s = overstress factor

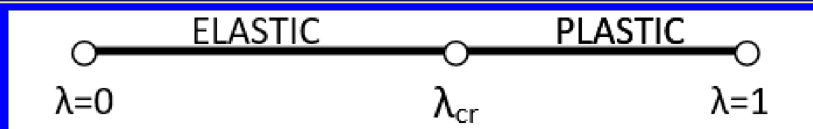
$$p = p_o(1-\lambda)$$

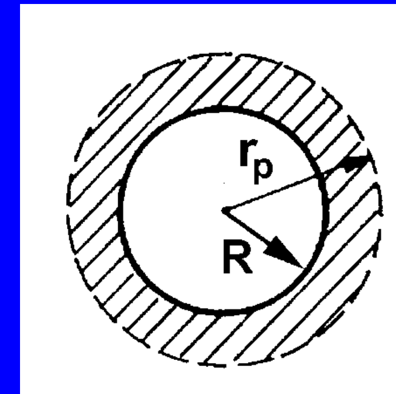
$$N_s = \frac{2\,p_o}{\sigma_{cm}}$$

Graph is valid for common values of the relevant parameters

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$





Ground displacement :

(a) Displacement (u_p) at the limit of the plastic zone ($r = r_p$) :

Calculated for a tunnel with radius $R=r_p$ and critical deconfinement (λ_{cr}), in which case ground displacements are elastic for $r > r_p$:

$$\frac{u_p}{R} = \lambda_{cr}\left(\frac{r_p}{R}\right)\left(\frac{p_o}{2G}\right) \qquad \lambda_{cr} = 1 - \left(\frac{2}{1+k}\right)\left(\frac{N_s - 1}{N_s}\right)$$

(b) Displacement (u) in the elastic zone ($r > r_p$) : $u = u_p\left(\frac{r_p}{r}\right)$

Calculated by the elastic formula:

$u = \frac{c_2}{r}$ with boundary condition: $u=u_p$ at $r=r_p$

(c) Displacement (u) in the plastic zone ($r < r_p$) : $\frac{u}{R} = \frac{u_p}{R}\left(\frac{r_p}{r}\right)^K$

and at the tunnel wall ($r = R$) : $\frac{u_R}{R} = \frac{u_p}{R}\left(\frac{r_p}{R}\right)^K$

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$

(d) Displacement ($u_{R\infty}$) at tunnel wall at full deconfinement (λ=1):

$$\frac{u_{R\infty}}{R} = \lambda_{cr}\left(\frac{p_o}{2G}\right)\left(\frac{r_{p\infty}}{R}\right)^{K+1}$$

$$\lambda_{cr} = 1 - \left(\frac{2}{1+k}\right)\left(\frac{N_s - 1}{N_s}\right)$$

$$\frac{r_{p\infty}}{R} = \left\{\left(\frac{1}{k+1}\right)\left[2 + N_s(k-1)\right]\right\}^{\frac{1}{k-1}}$$

i.e.,:

$$\frac{u_{R\infty}}{R} = f\left(\frac{p_o}{2G}, N_s, \phi, \delta\right)$$

Displacement (u_R) at tunnel wall, for any deconfinement $\lambda > \lambda_{cr}$):

$$\frac{u_R}{u_{R\infty}} = \left\{\frac{1}{1 + \frac{N_s}{2}(k-1)(1-\lambda)}\right\}^{\left(\frac{K+1}{k-1}\right)} = f(\lambda\ ; N_s, \phi, \delta)$$

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$

Proof of formulae for u_r :

B.3 Υπολογισμός των μετακινήσεων στην πλαστική ζώνη ($r < r_p$) :

Ορισμός διαστολικότητας στην πλαστική ζώνη : $\tan\delta = \dfrac{\varepsilon_r + \varepsilon_\theta}{\varepsilon_r - \varepsilon_\theta} \geq 0$

οπότε : $K \equiv \dfrac{1+\tan\delta}{1-\tan\delta} = -\dfrac{\varepsilon_r}{\varepsilon_\theta} \geq 1$

Αλλά : $\varepsilon_r = \dfrac{du}{dr}$, $\varepsilon_\theta = \dfrac{u}{r}$

Οπότε: $\varepsilon_\theta K + \varepsilon_r = 0 \Rightarrow \dfrac{u}{r}K + \dfrac{du}{dr} = 0 \Rightarrow u = \alpha \dfrac{1}{r^K}$

Συνοριακή συνθήκη: $r = r_p \Rightarrow u = u_p \Rightarrow u = u_p\left(\dfrac{r_p}{r}\right)^K$

Αλλά το u_p έχει υπολογισθεί από την ελαστική ζώνη. Συνεπώς :

(α) Περίπτωση $k \neq 1 \Rightarrow \varphi \neq 0$:

$$u = u_p\left(\frac{r_p}{r}\right)^K \Rightarrow \frac{u_R}{R} = \frac{u_p}{R}\left(\frac{r_p}{R}\right)^K$$

όπου :

$$u_p = r_p\left(\frac{p_o}{2G}\right)\left(1-\frac{2}{k+1}\right)\left[1+\frac{2}{(k-1)N_s}\right] \Rightarrow u_p = r_p\left(\frac{p_o}{2G}\right)\frac{(k-1)N_s+2}{(k+1)N_s}$$

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$

Proof of formulae for u_r :

και : $$\frac{r_p}{R} = \left[\left(\frac{2}{k+1} \right) \frac{N_s + \frac{2}{k-1}}{(1-\lambda)N_s + \frac{2}{k-1}} \right]^{\frac{1}{k-1}}$$

Για λ = 1 :

$$\frac{r_{p\infty}}{R} = \left[\frac{(k-1)N_s + 2}{k+1} \right]^{\frac{1}{k-1}} \quad \text{και} \quad \frac{u_{p\infty}}{R} = \frac{r_{p\infty}}{R} \left(\frac{p_o}{2G} \right) \frac{(k-1)N_s + 2}{(k+1)N_s}$$

Προσδιορισμός της τελικής (για λ=1) σύγκλισης του τοιχώματος της σήραγγας ($u_{R\infty}$) :

$$\frac{u_{R\infty}}{R} = \frac{u_{p\infty}}{R} \left(\frac{r_{p\infty}}{R} \right)^K \Rightarrow \frac{u_{R\infty}}{R} = \frac{1}{N_s} \left(\frac{p_o}{2G} \right) \left[\frac{(k-1)N_s + 2}{k+1} \right]^{\frac{K+k}{k-1}}$$

Παρατήρηση :

Επειδή στην ελαστική περίπτωση η τελική (για λ=1) σύγκλιση του τοιχώματος της σήραγγας ($u_{R\infty,e}$) είναι : $\frac{u_{R\infty,e}}{R} = \left(\frac{p_o}{2G} \right)$ προκύπτει ότι :

$$\frac{u_{R\infty}}{u_{R\infty,e}} = \frac{1}{N_s} \left[\frac{(k-1)N_s + 2}{k+1} \right]^{\frac{K+k}{k-1}} \quad \text{ή} \quad \frac{u_{R\infty}}{u_{R\infty,e}} = \frac{1}{N_s} \left[\frac{r_{p\infty}}{R} \right]^{K+k}$$

CASE 2: Plastic zone develops in the ground

If $N_s > 1$ and $\lambda > \lambda_{cr}$

Proof of formulae for u_r:

(β) Περίπτωση $k=1 \Rightarrow \varphi = 0$:

$$u = u_p\left(\frac{r_p}{r}\right)^K \Rightarrow \frac{u_R}{R} = \frac{u_p}{R}\left(\frac{r_p}{R}\right)^K$$

όπου: $u_p = r_p\left(\frac{p_o}{2G}\right)\frac{1}{N_s}$

και : $r_p = R\exp\left[\frac{1}{2}(N_s\lambda - 1)\right]$ και $r_{p\infty} = R\exp\left[\frac{1}{2}(N_s - 1)\right]$

Προσδιορισμός της τελικής (για λ=1) σύγκλισης του τοιχώματος της σήραγγας ($u_{R\infty}$) :

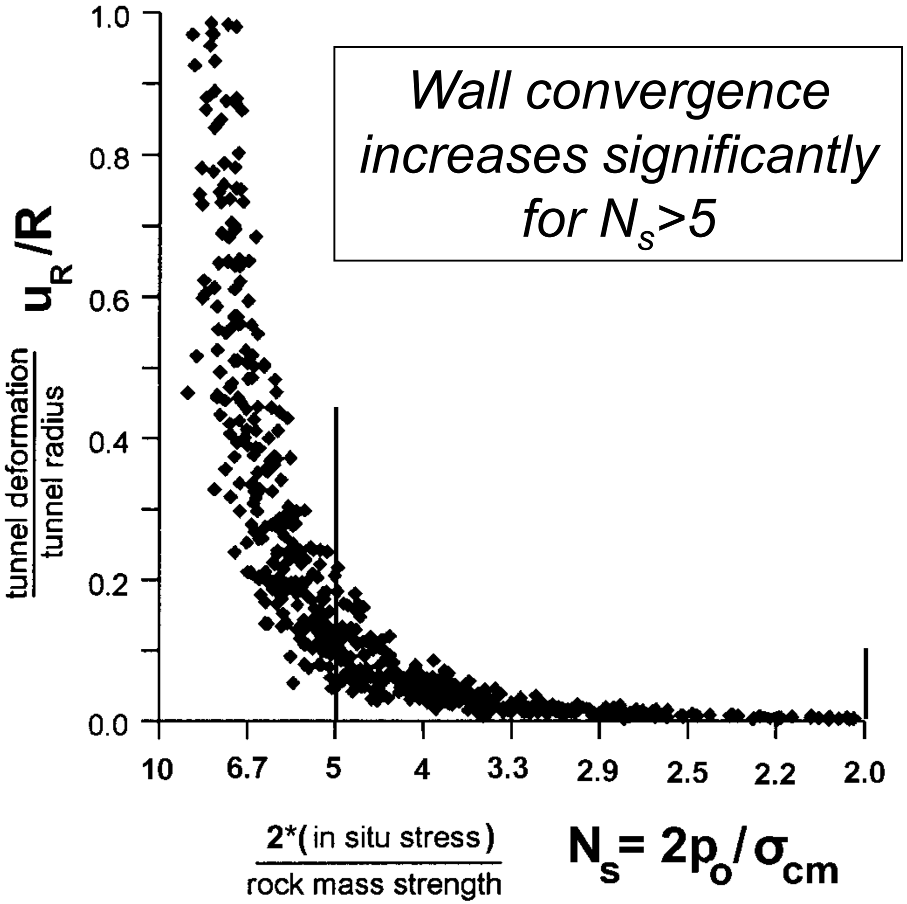
$$\frac{u_{R\infty}}{R} = \frac{u_{p\infty}}{R}\left(\frac{r_{p\infty}}{R}\right)^K \Rightarrow$$

$$\frac{u_{R\infty}}{R} = \frac{1}{N_s}\left(\frac{p_o}{2G}\right)\exp\left[\frac{1}{2}(N_s - 1)(K+1)\right]$$

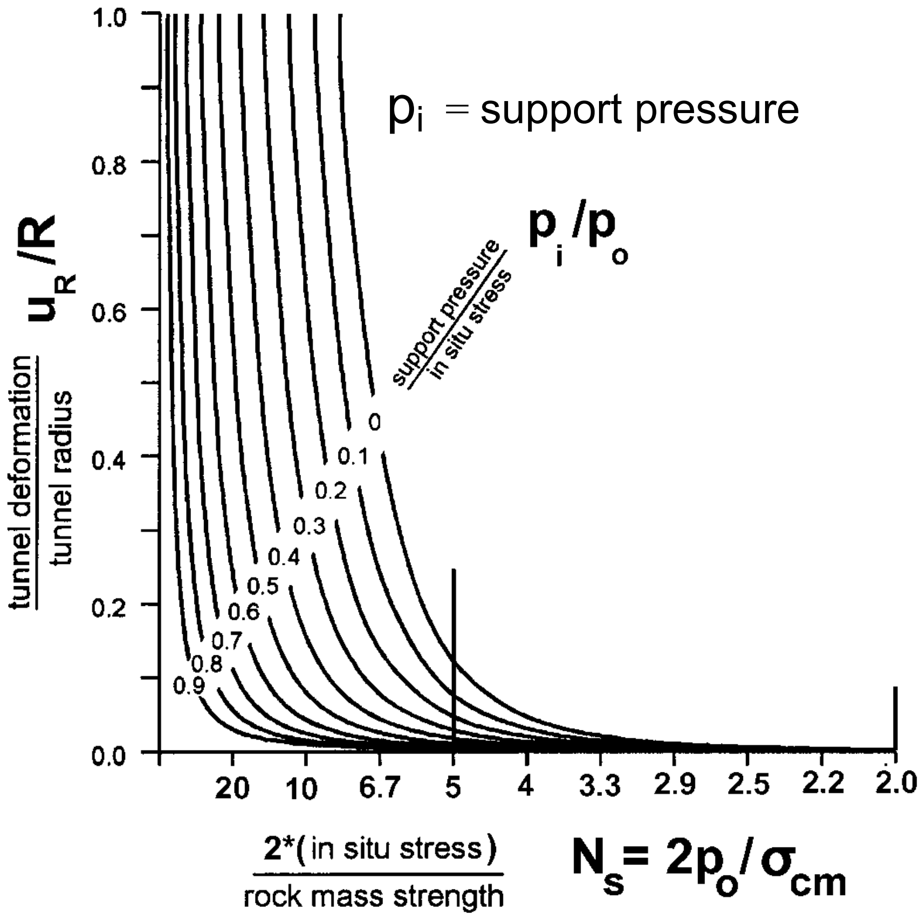
δηλαδή : $\frac{u_{R\infty}}{u_{R\infty,e}} = \frac{1}{N_s}\exp\left[\frac{1}{2}(N_s - 1)(K+1)\right]$ ή $\frac{u_{R\infty}}{u_{R\infty,e}} = \frac{1}{N_s}\left[\frac{r_{p\infty}}{R}\right]^{K+1}$

Displacement of tunnel wall (u_R)

Unsupported tunnel



Supported tunnel



- *Data points for common values of the relevant parameters*
- *Significant reduction of wall displacement with increasing support pressure*

Stresses at the tunnel wall (r=R)

1. Elastic ground ($\lambda < \lambda_{cr}$): $\dfrac{\sigma_r}{p_o} = (1-\lambda)$ $\dfrac{\sigma_\theta}{p_o} = (1+\lambda)$

2. Elasto-plastic ground ($\lambda > \lambda_{cr}$): $\dfrac{\sigma_r}{p_o} = (1-\lambda)$ $\dfrac{\sigma_\theta}{p_o} = k\left(\dfrac{\sigma_r}{p_o}\right) + \dfrac{2}{N_s}$

Proof:

Equilibrium equation: $$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

Mohr-Coulomb criterion: $$\sigma_\theta = k\sigma_r + \sigma_{cm}$$

Combining the above: $$\frac{d\sigma_r}{dr} - \frac{1}{r}(k-1)\sigma_r - \frac{1}{r}\sigma_{cm} = 0$$

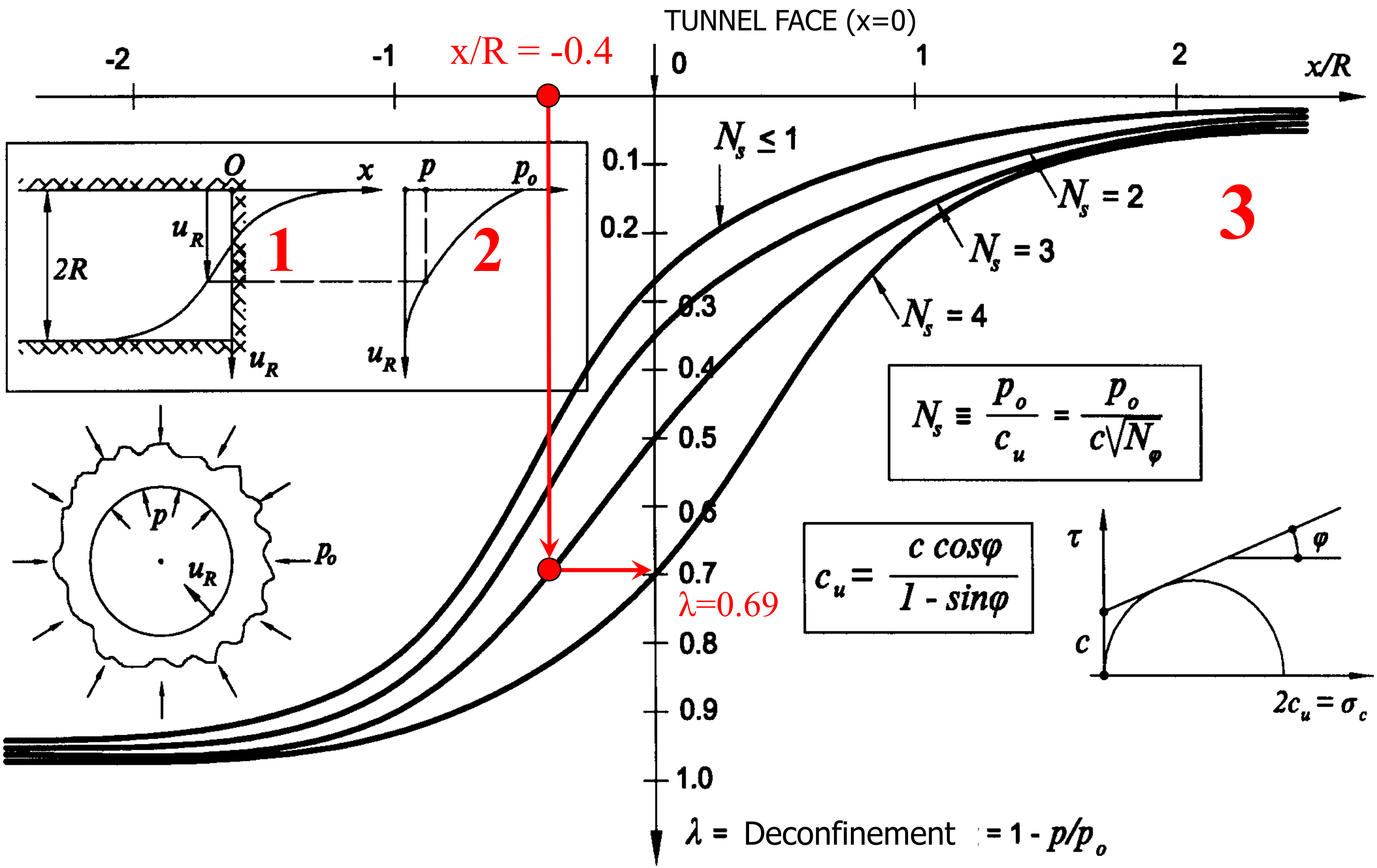
Boundary condition: $$\sigma_r(r=R) = (1-\lambda)\,p_o$$

Solution for $\phi \neq 0 \Rightarrow$ $$\sigma_r = \left[(1-\lambda)\,p_o + \left(\frac{\sigma_{cm}}{k-1}\right)\right]\left(\frac{r}{R}\right)^{k-1} - \left(\frac{\sigma_{cm}}{k-1}\right)$$

Solution for $\phi = 0 \Rightarrow$ $$\sigma_r = (1-\lambda)\,p_o + \sigma_{cm}\ln\left(\frac{r}{R}\right)$$

Combining the Chern u_R-x curve (1), with the convergence-confinement u_R-p curve (2), one can develop the Chern-Panet curves λ-x (3).

The Chern-Panet curves are useful in 3D numerical analyses (to compute λ from x)



Chern-Panet curves

1. Displacement u_R at tunnel wall as a function of deconfinement (λ):

For deconfinement $\lambda > \lambda_{cr}$ (plasticity):

$$\frac{u_R(\lambda)}{u_{R\infty}} = \left\{ \frac{1}{1 + \frac{N_s}{2}(k-1)(1-\lambda)} \right\}^{\left(\frac{K+1}{k-1}\right)} = f(\lambda \;; N_s, \phi, \delta)$$

For deconfinement $\lambda < \lambda_{cr}$ (elasticity):

$$\frac{u_R(\lambda)}{u_{R\infty}} = \lambda$$

2. Displacement u_R at tunnel wall along the tunnel axis (x) (Chern, 1998) :

$$\frac{u_R(x)}{u_{R\infty}} = \left[1 + exp\left(0.91 \frac{x}{R} \right) \right]^{-1.7}$$

Combination of (1), (2) gives the Chern-Panet curves, in the form:

$$\lambda = f\left(\frac{x}{R};\; N_s, \varphi, \delta \right)$$

These curves calculate the deconfinement coefficient (λ) at any location (x) along the tunnel axis. They are used in numerical analyses for the calculation of (λ) at the location (x) of support application

Example:

Tunnel radius D = 6m - tunnel depth H = 100m

γ = 22 kN/m^3, K_o = 0.60 $\Rightarrow$ $p_o = 0.5\,(1+K_o)\,\gamma\,H$ = 1.76 MPa

GSI = 25, σ_{ci} = 12 MPa, E_i = 13.5 GPa $\Rightarrow$ σ_{cm} = 0.64 MPa , E = 821 MPa

$$\sigma_{cm} = \frac{\sigma_{ci}}{52.63}\exp\left(\frac{GSI}{20}\right)$$

$$E_{rm} = E_i\left(0.02 + \frac{1 - D/2}{1 + e^{((60+15D-GSI)/11)}}\right)$$

D = damage factor (=0)

ν = 0.30 $\Rightarrow$ G = 316 MPa $\quad G = \dfrac{E}{2(1+\nu)}$

φ = 32^o $\Rightarrow$ k = 3.2546 $\quad k = \tan^2\left(45 + \dfrac{\varphi}{2}\right)$

δ = 7^o $\Rightarrow$ K = 1.28 $\quad K \equiv \dfrac{1+\tan\delta}{1-\tan\delta}$

Calculations:

$$N_s = \frac{2\,p_o}{\sigma_{cm}} = 5.5 \qquad \lambda_{cr} = 1 - \left(\frac{2}{1+k}\right)\left(\frac{N_s - 1}{N_s}\right) = 0.615$$

$$\frac{r_{p\infty}}{R} = \left\{\left(\frac{1}{k+1}\right)\left[2 + N_s(k-1)\right]\right\}^{\frac{1}{k-1}} = 1.72 \qquad \frac{u_{R\infty}}{R} = \lambda_{cr}\left(\frac{p_o}{2G}\right)\left(\frac{r_{p\infty}}{R}\right)^{K+1}$$

$$\frac{u_{R\infty}}{R} = 0.00588 \Rightarrow u_{R\infty} = 600 \times 0.00588 = 3.53\ cm$$

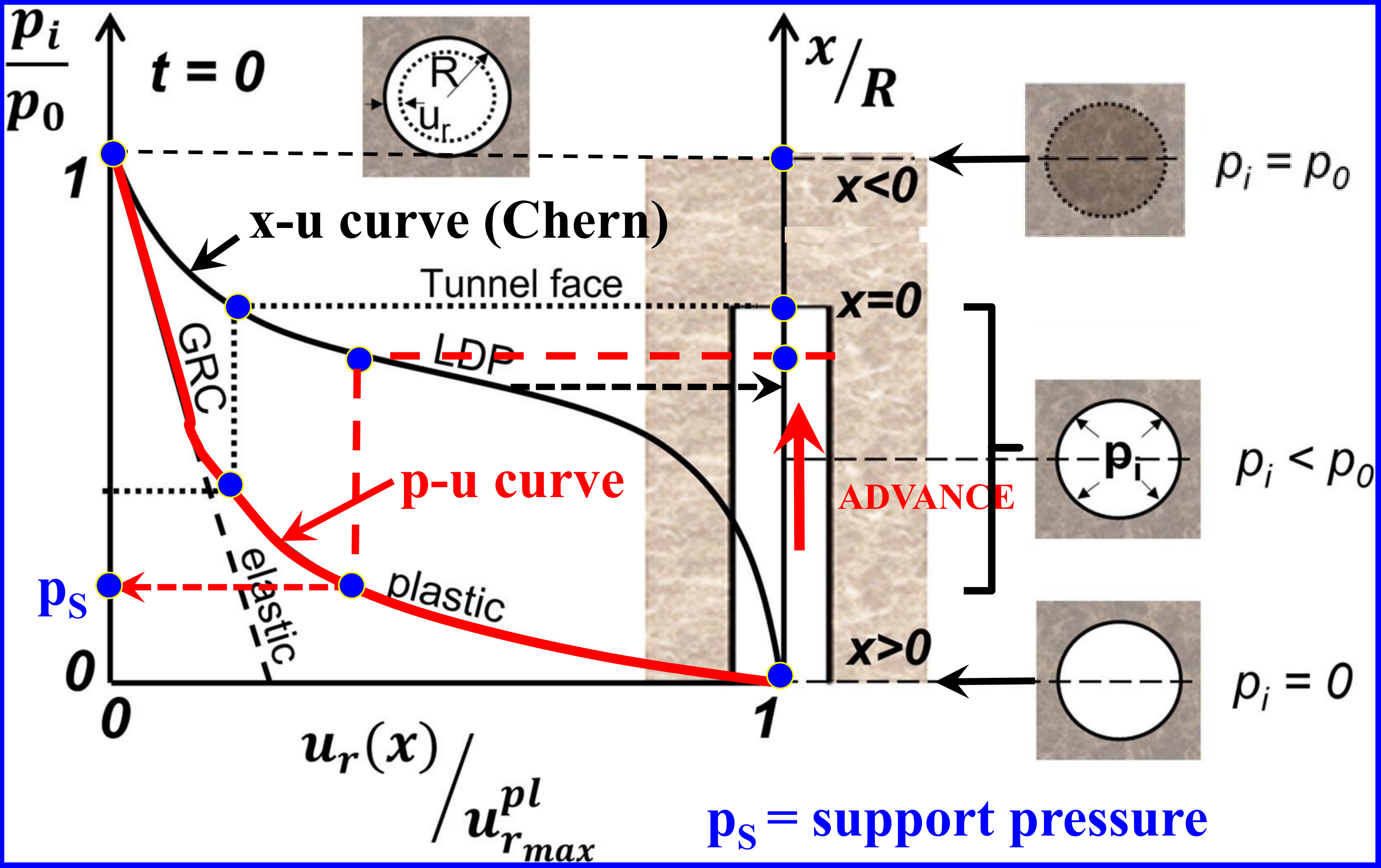
Ground Reaction and Longitudinal Displacement (Chern) curves

Convergence - Confinement (or Ground Reaction) curve:

GRC = Ground Reaction Curve: $U_r(x) / U_{r,max}$ versus p_i / p_o

LDP = Longitudinal Displacement Profile or Chern curve: $U_r(x) / U_{r,max}$ versus x / R

Combination of the GRC and LDP curves provides the relation: p_i / p_o versus x / R which is required in 2D numerical analyses.



EXCEL spreadsheet for the calculation of the GRC and LDP curves

Input data: R , p_o , σ_{cm} , φ , δ , G

$$N_s = \frac{2\,p_o}{\sigma_{cm}} \qquad k = \tan^2\left(45 + \frac{\phi}{2}\right)$$

Calculate N_s , k , K and λ_{cr}

$$K \equiv \frac{1+\tan\delta}{1-\tan\delta} \qquad \lambda_{cr} = 1 - \left(\frac{2}{1+k}\right)\left(\frac{N_s - 1}{N_s}\right)$$

Col 1: p/p_o between 1 ... 0

Col 2: λ (between 0 ... 1)

$$\lambda = 1 - \frac{p}{p_o}$$

Col 3: Plastic region ? (Y/N) ⟶ If $\lambda > \lambda_{cr}$ then Y else N

Col 4: r_p /R ⟶ If $\lambda < \lambda_{cr}$ (no plastic region) then $r_p/R = 1$ else:

If $\varphi = 0$: $$\frac{r_p}{R} = \exp\left[\frac{1}{2}(\lambda\, N_s - 1)\right]$$

If $\varphi > 0$: $$\frac{r_p}{R} = \left\{\left(\frac{2}{k+1}\right)\left[\frac{2 + N_s(k-1)}{2 + N_s(k-1)(1-\lambda)}\right]\right\}^{\frac{1}{k-1}}$$

Col 5: u_p /R ⟶ If $\lambda < \lambda_{cr}$ then u_p = n/a else: $$\frac{u_p}{R} = \lambda_{cr}\left(\frac{r_p}{R}\right)\left(\frac{p_o}{2G}\right)$$

EXCEL spreadsheet for the calculation of the convergence – confinement curve

Col 6: u_R/R ⟶ If $\lambda < \lambda_{cr}$ (no plastic region): $\dfrac{u_R}{R} = \lambda\left(\dfrac{p_o}{2G}\right)$

else: $\dfrac{u_R}{R} = \dfrac{u_p}{R}\left(\dfrac{r_p}{R}\right)^K$

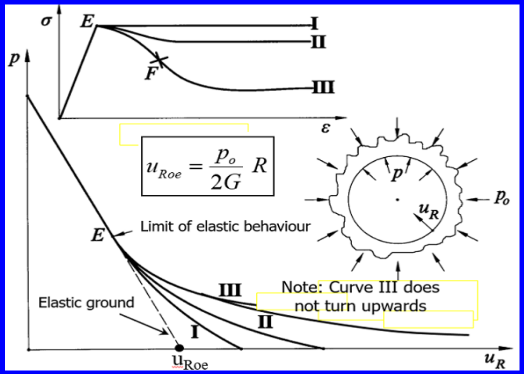
Calculate $u_{R\infty}/R$: equal to u_R/R for $\lambda=1$

Col 7: $u_R / u_{R\infty}$ ⟶ $(u_R/R) / (u_{R\infty}/R)$

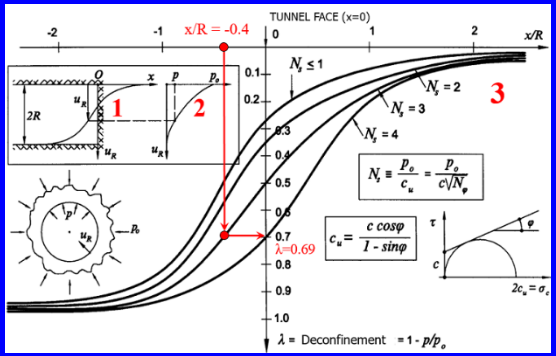
Col 8: x / R ⟶ $\dfrac{x}{R} = 1.10\ \ln\left[\left(\dfrac{u_R}{u_{R\infty}}\right)^{-0.588} - 1\right]$

Plot curves: (u_R/R) vs (p/p_o) , (r_p/R) vs (p/p_o) , (x/R) vs (p/p_o) or (u_R/R)

Ground Reaction (GRC)



Longitudinal Diplacement (LDP)



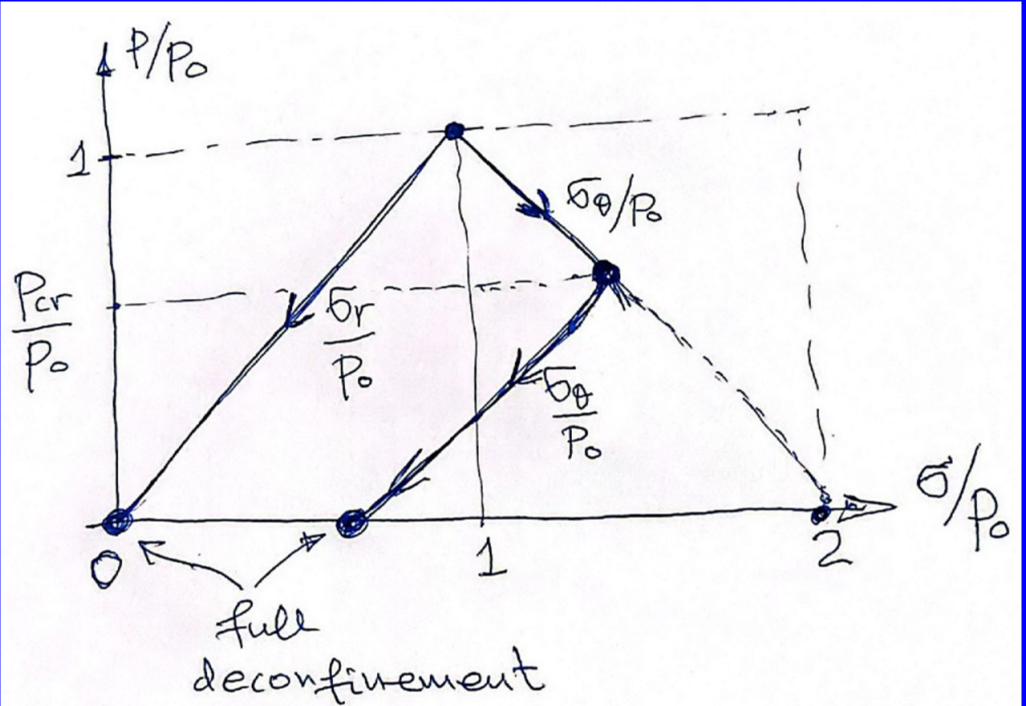
EXCEL spreadsheet for the calculation of the convergence – confinement curve

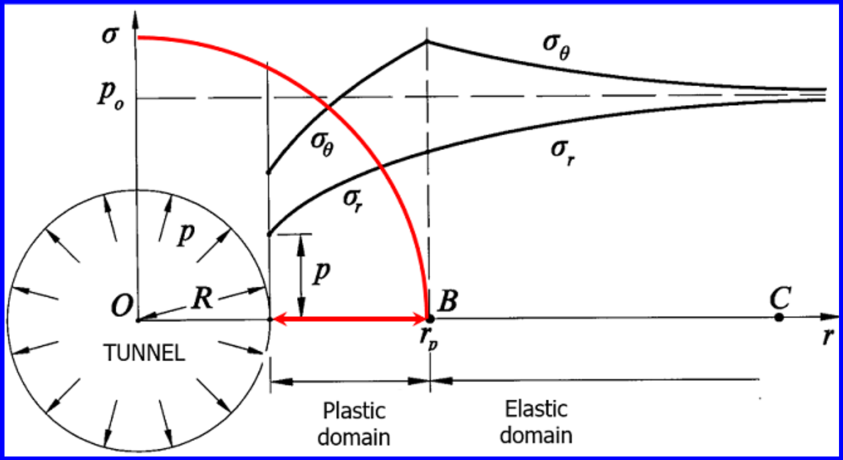
Col 9: σ_r / p_o (at r=R): $\longrightarrow$ $\dfrac{\sigma_r}{p_o} = (1-\lambda)$

Col 10: σ_θ / p_o (at r=R): $\longrightarrow$ If $\lambda < \lambda_{cr}$ then: $\dfrac{\sigma_\theta}{p_o} = (1+\lambda)$

else: $\dfrac{\sigma_\theta}{p_o} = k\left(\dfrac{\sigma_r}{p_o}\right) + \dfrac{2}{N_s}$

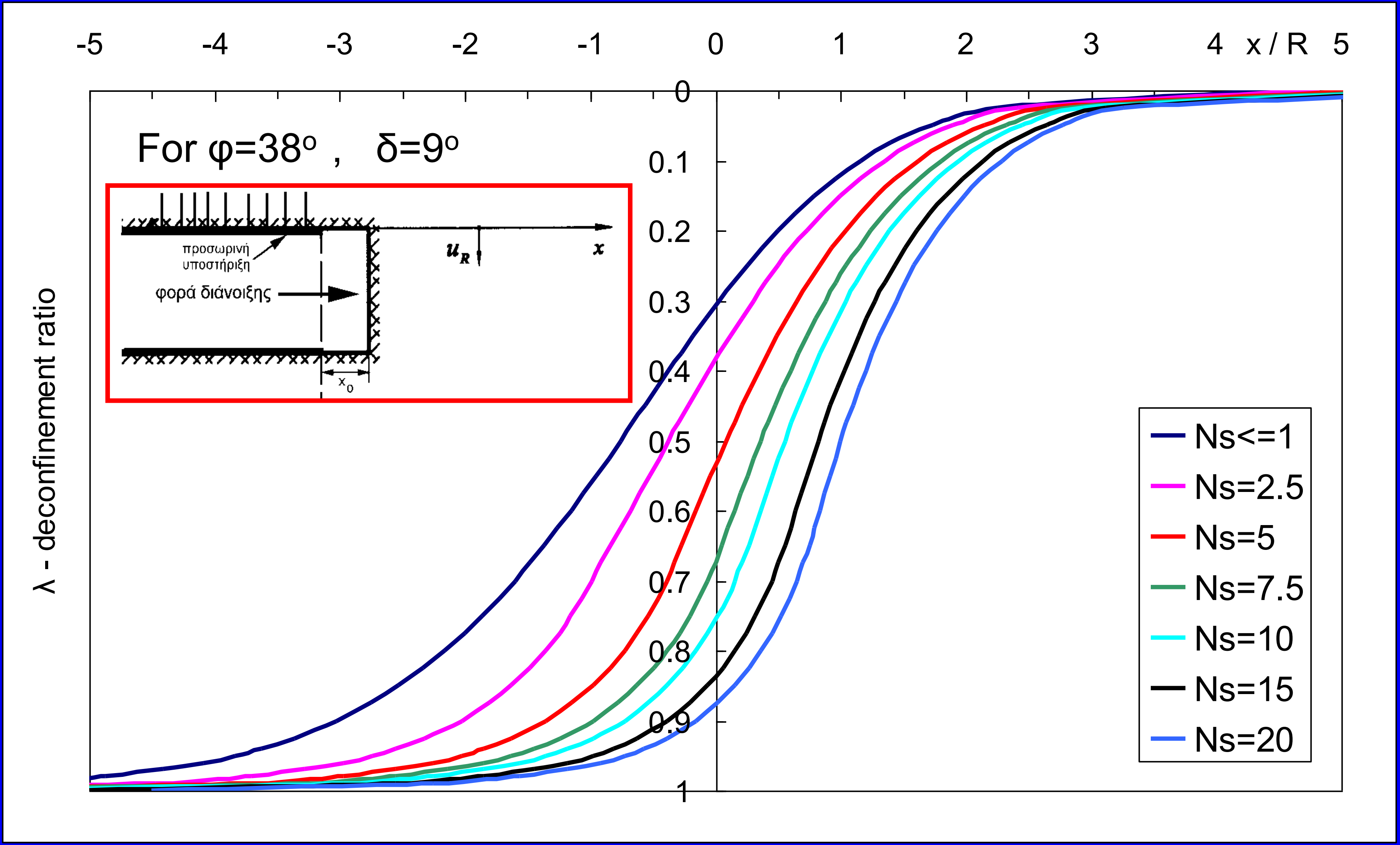
Plot curves: (σ_r / p_o & σ_θ / p_o) vs (p/p_o)





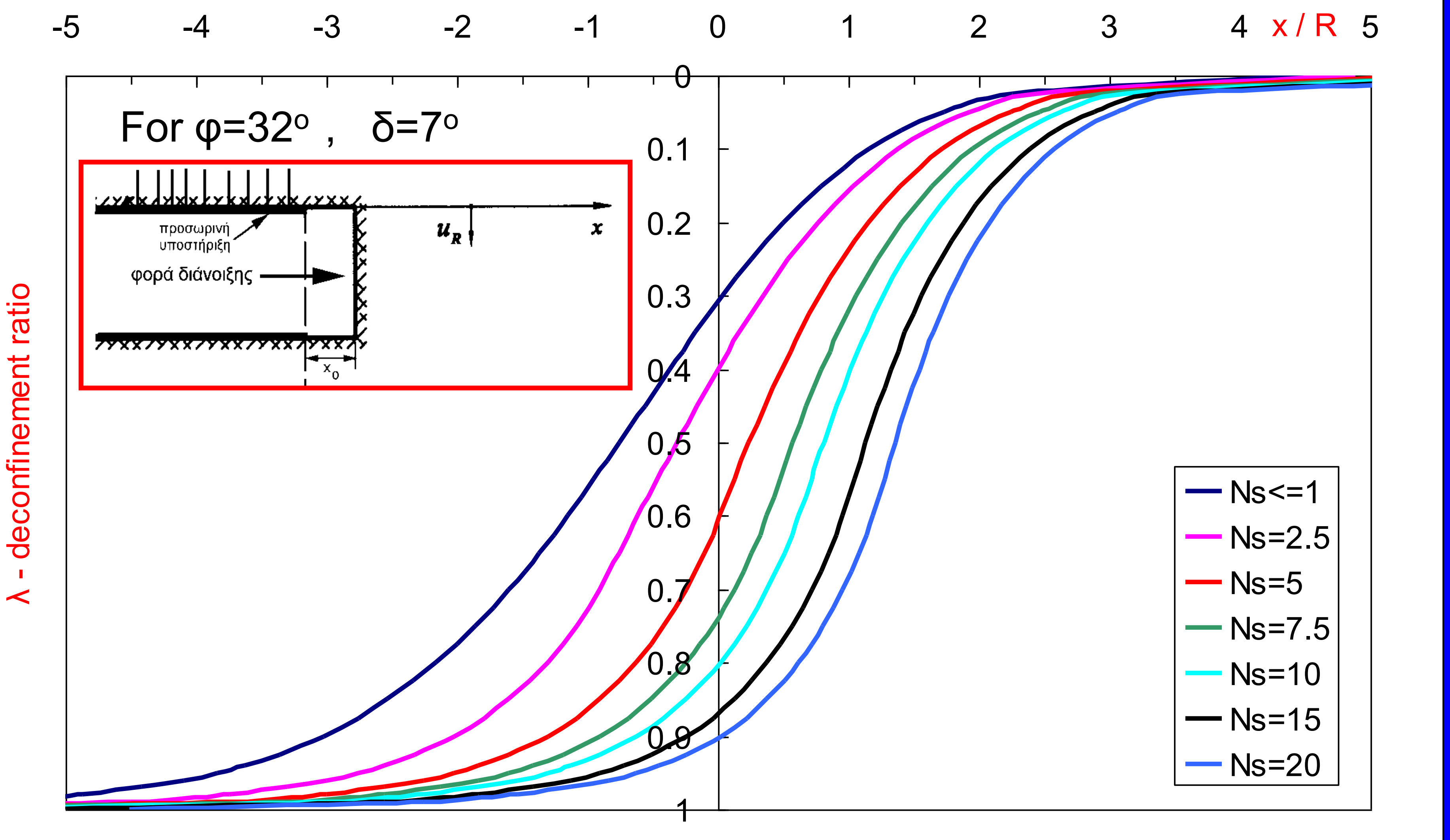
Examples of Panet – Chern curves :

$$\lambda = f\left(\frac{x}{R};\ N_s, \varphi, \delta\right)$$

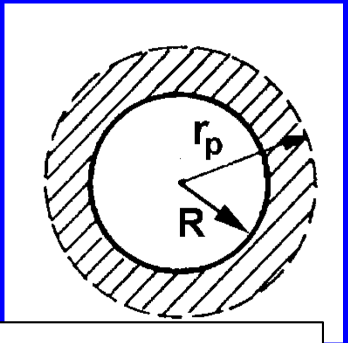


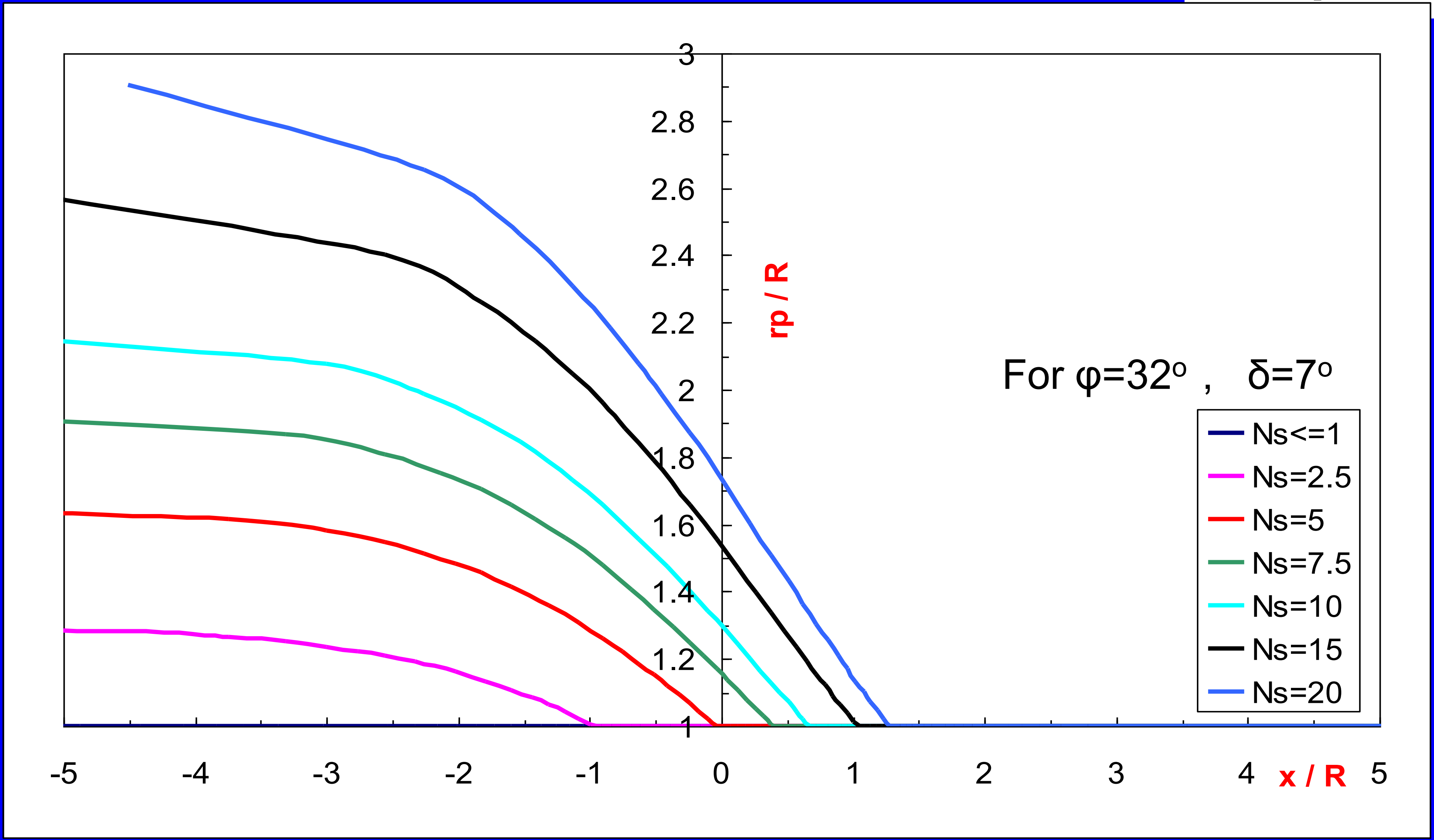
Examples of Panet – Chern curves :

$$\lambda = f\left(\frac{x}{R};\ N_s, \varphi, \delta\right)$$

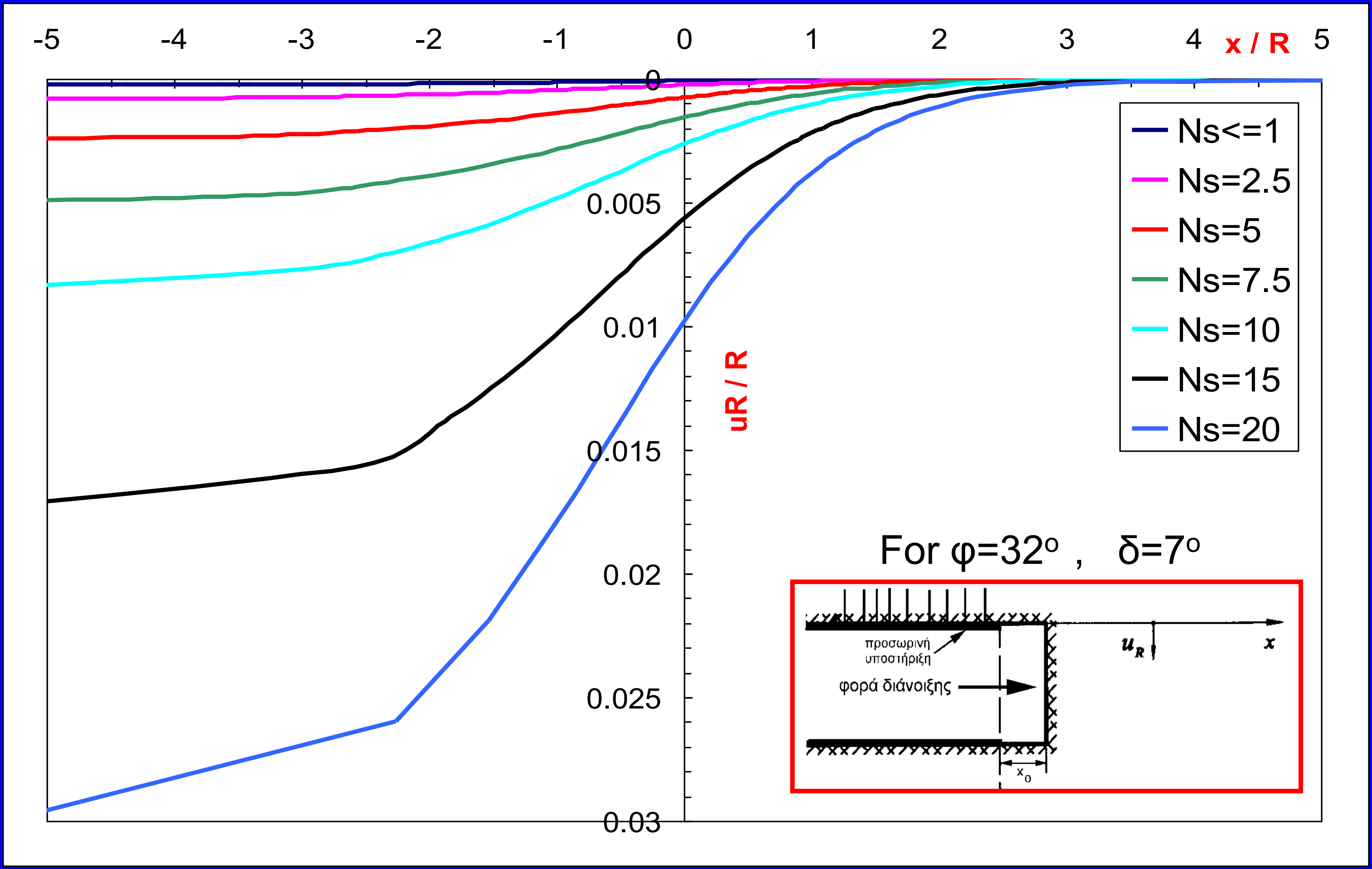


Examples of radius of plastic zone: $\frac{r_P}{R} = f(\lambda \; ; \; N_s, \phi)$





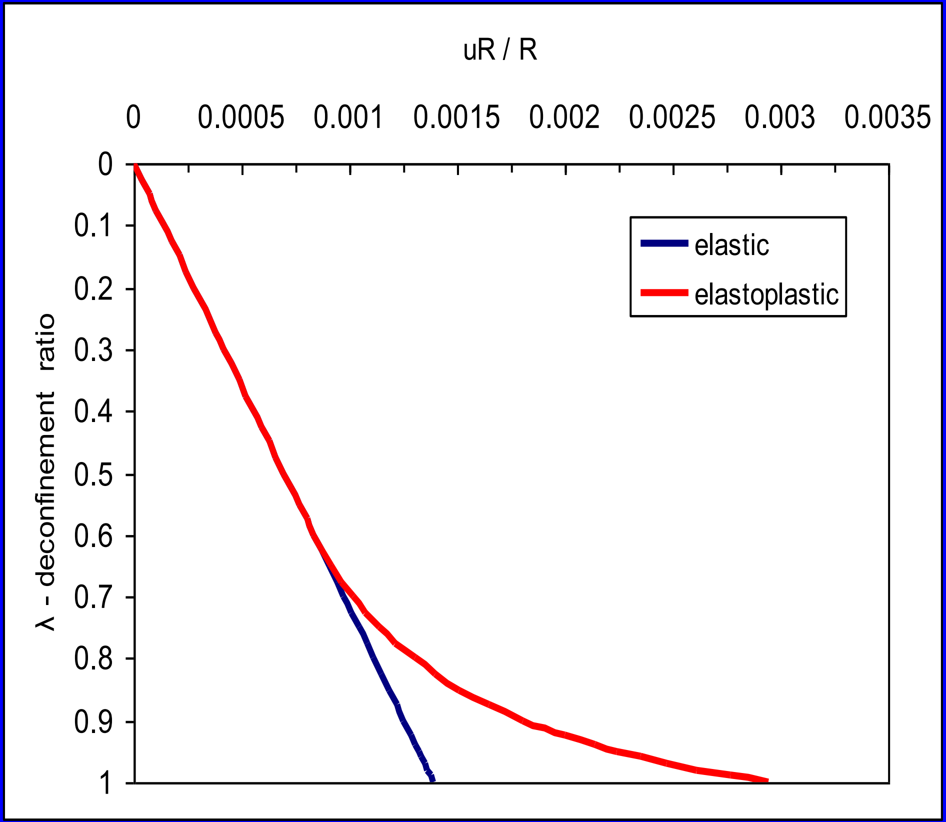
Examples of tunnel wall displacement: $\frac{u_R}{R} = f\left(\frac{x}{R}, \frac{p_o}{2G}\ ;\ N_s, \varphi, \delta\right)$

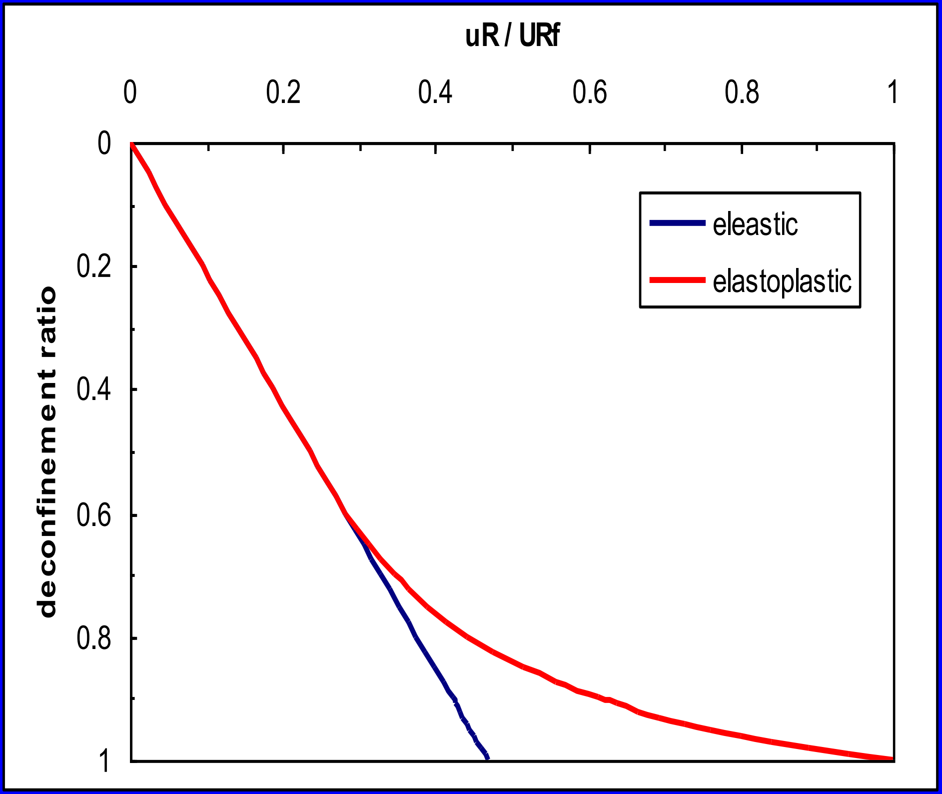


Example: Convergence – confinement curve (u_R) - (λ)

$$\frac{u_R}{R} = f\left(\lambda, \frac{p_o}{2G} \; ; \; N_s, \varphi, \delta\right)$$

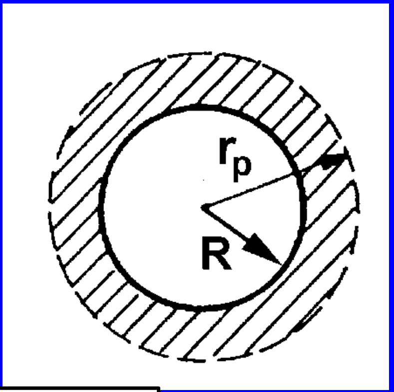
$$\frac{u_R}{u_{R\infty}} = f\left(\lambda \; ; \; N_s, \varphi, \delta\right)$$

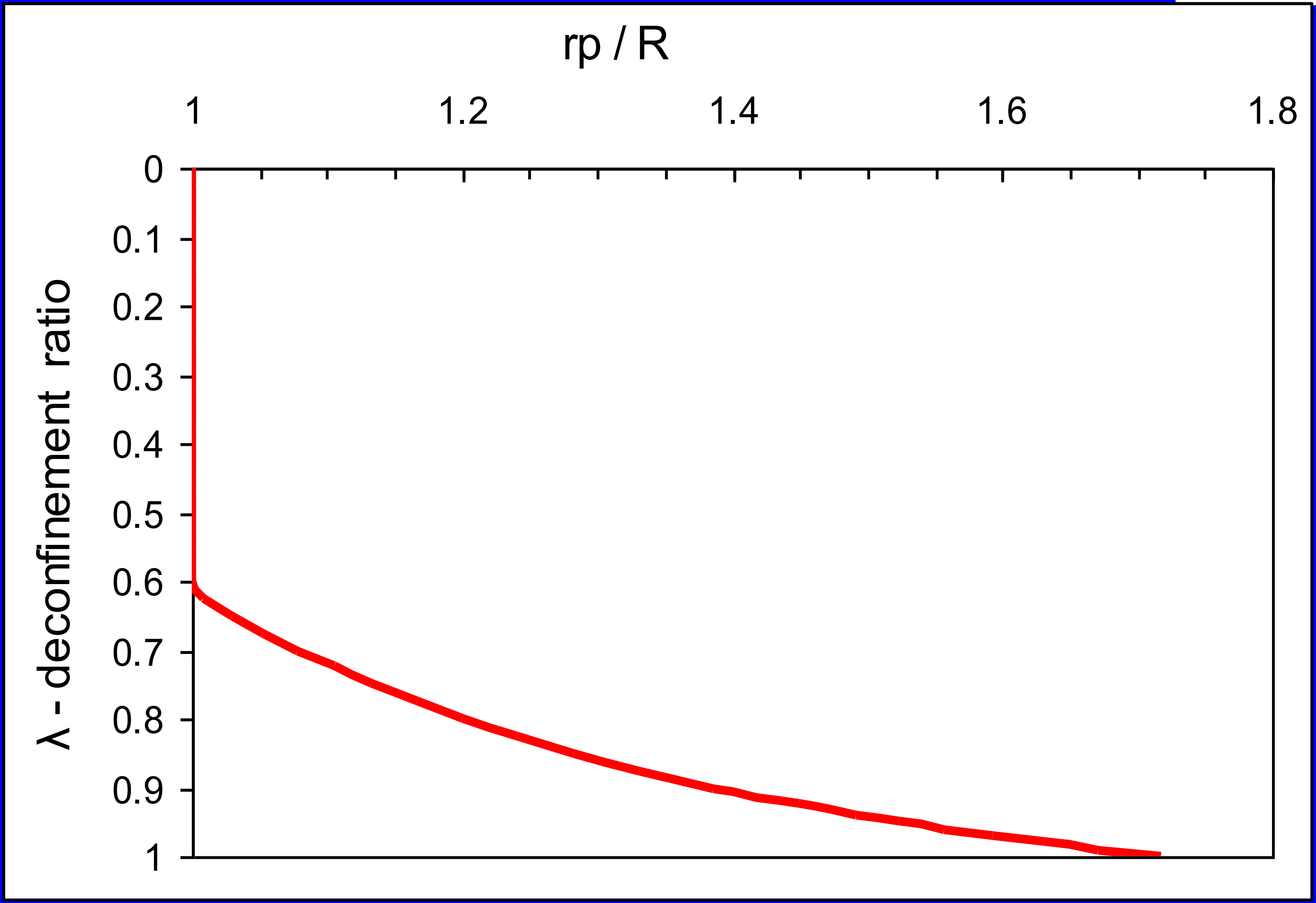




Example: Radius of the plastic zone

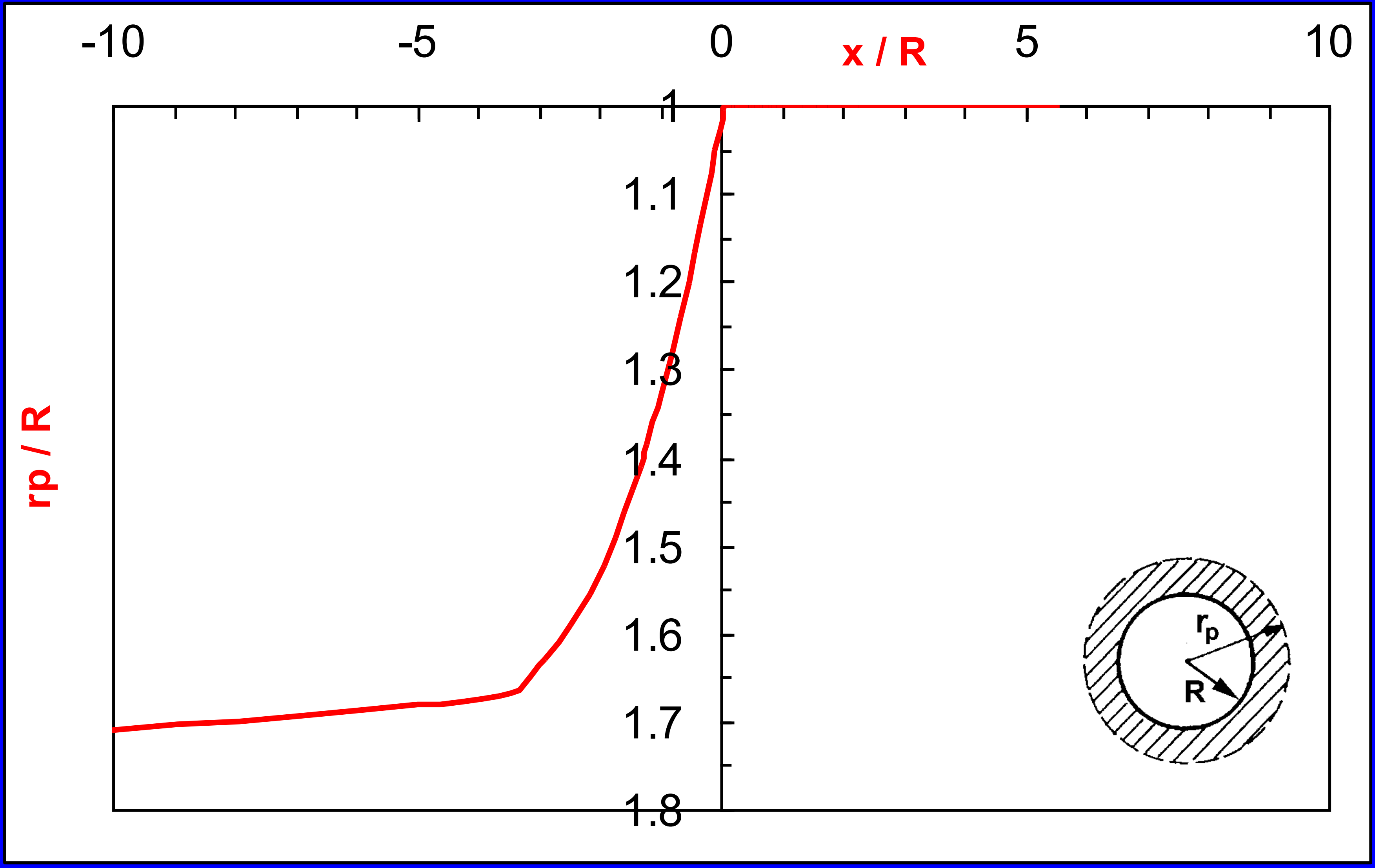
$$\frac{r_P}{R} = f\left(\lambda \; ; \; N_s \, , \phi \right)$$





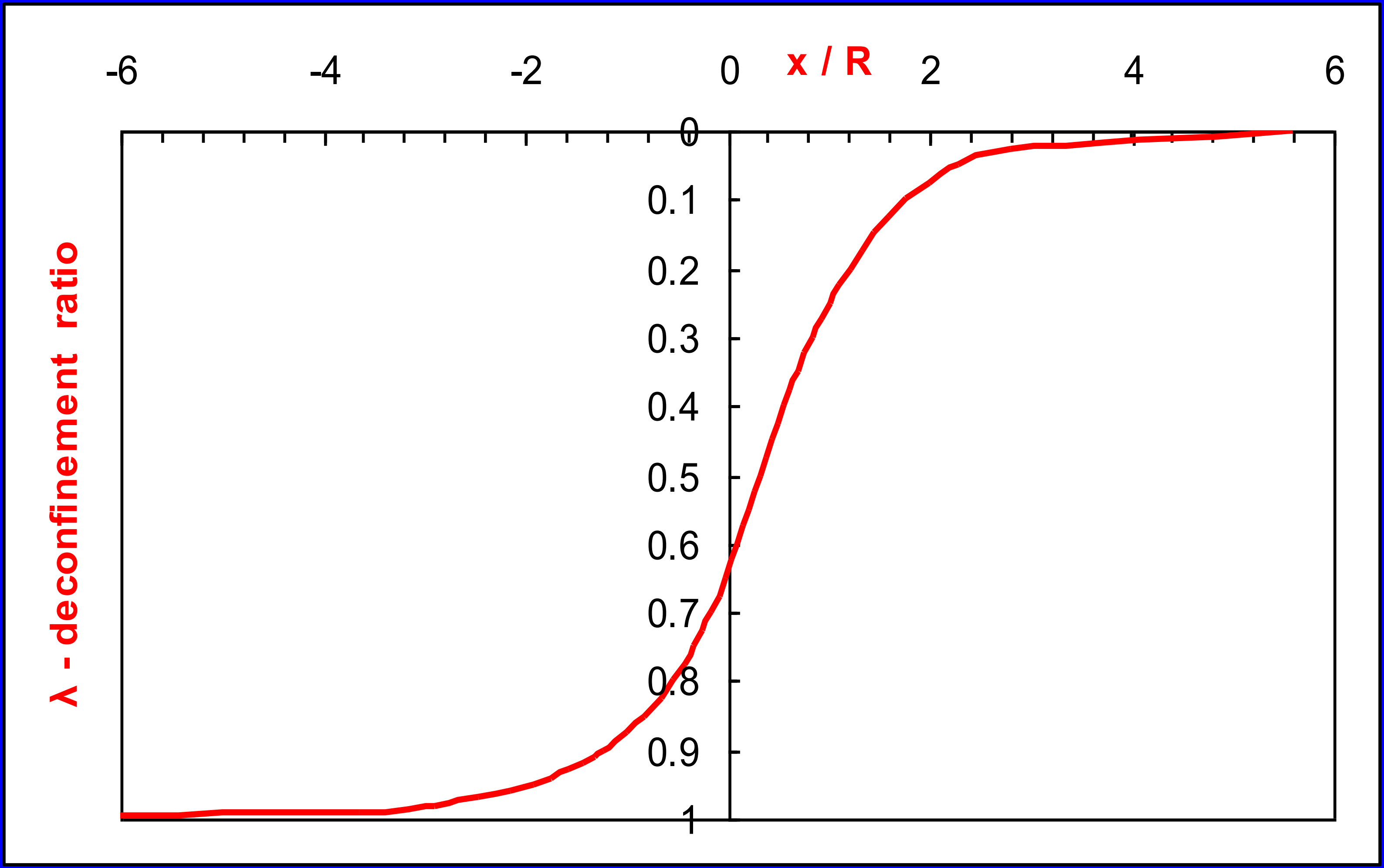
Example: Radius of plastic zone

$$\frac{r_P}{R} = f\left(\frac{x}{R};\ N_s, \varphi, \delta\right)$$



Example: Panet - Chern curve

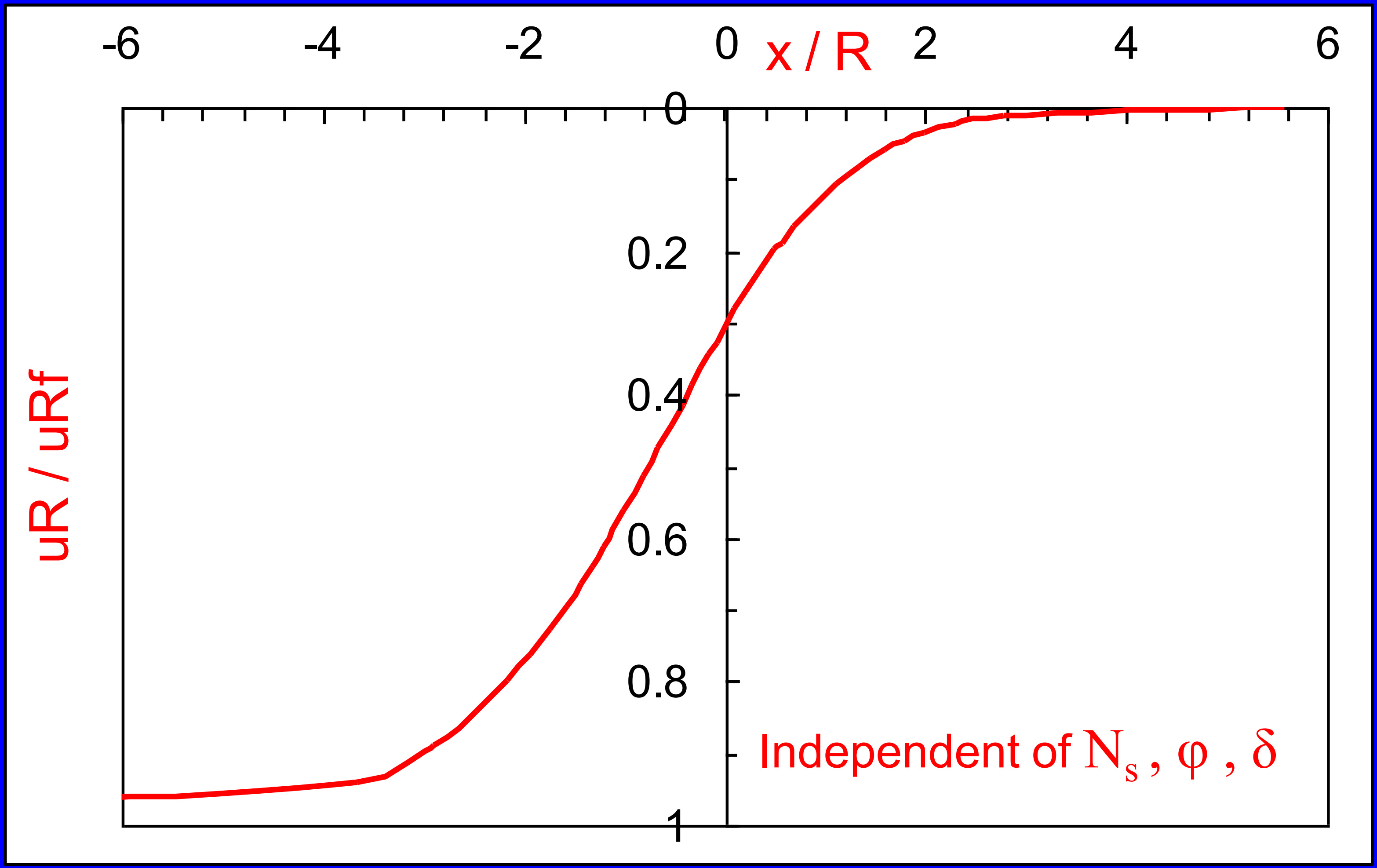
$$\frac{x}{R} = 1.10 \ \ln\left[\left(\frac{u_R(x)}{u_{R\infty}}\right)^{-0.588} - 1\right] \Rightarrow \lambda = f\left(\frac{x}{R};\ N_s, \varphi, \delta\right)$$



Example: wall displacement curve (A)

$$\frac{u_R}{u_{R\infty}} = f\left(\frac{x}{R}\right) \qquad \text{(Chern)}$$

$$\frac{u_R(x)}{u_{R\infty}} = \left[1 + exp\left(0.91\frac{x}{R}\right)\right]^{-1.7}$$



Example: wall displacement curve (B)

$$\frac{u_R}{R} = f\left(\frac{x}{R}, \frac{p_o}{2G} \; ; \; N_s, \varphi, \delta\right)$$