Part B TORSION OF INELASTIC CIRCULAR BARS

6-13. Shear Stresses and Deformations in Circular Shafts in the Inelastic Range

The torsion formula for circular sections previously derived is based on Hooke's law. Therefore, it applies only up to the point where the proportional limit of a material in shear is reached in the outer annulus of a shaft. Now the solution will be extended to include inelastic behavior of a material. As before, the equilibrium requirements at a section must be met. The deformation assumption of linear strain variation from the axis remains applicable. Only the difference in material properties affects the solution.

A section through a shaft is shown in Fig. 6-28(a). The linear strain variation is shown schematically in the same figure. Some possible mechanical properties of materials in shear, obtained, for example, in experiments with thin tubes in torsion, are shown in Figs. 6-28(b), (c), and (d). The corresponding shear stress distribution is shown to the right in each case. The stresses are determined from the strain. For example, if the shear strain is *a* at an interior annulus, Fig. 6-28(a), the corresponding stress is found from the stress-strain diagram. This procedure is appliable to solid shafts as well as to integral shafts made of concentric tubes of different materials, provided that the corresponding stress-strain diagrams are used. The derivation for a linearly elastic material is simply a special case of this approach.

After the stress distribution is known, torque T carried by these stresses is found as before; that is,

$$T = \int_A (\tau \, dA)\rho \qquad (6\text{-}27)$$

This integral must be evaluated over the cross-sectional area of the shaft.

Although the shear stress distribution after the elastic limit is exceeded is nonlinear and the elastic torsion formula, Eq. 6-3, does not apply, it is sometimes used to calculate a fictitious stress for the ultimate torque. The computed stress is called the *modulus of rupture*; see the largest ordinates of the dashed lines in Figs. 6-28(f) and (g). It serves as a rough index of the ultimate strength of a material in torsion. For a thin-walled tube, the stress distribution is very nearly the same regardless of the mechanical properties of the material; see Fig. 6-29. For this reason, experiments with thin-walled tubes are widely used in establishing the shear stress-strain τ-γ diagrams.

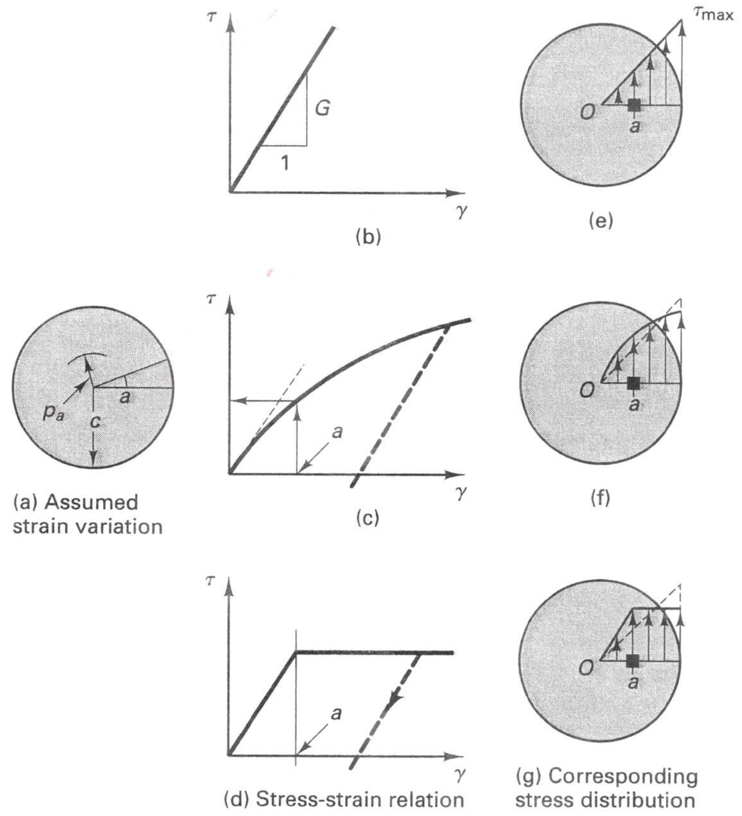

Fig. 6-28 Stresses in circular members due to torque.

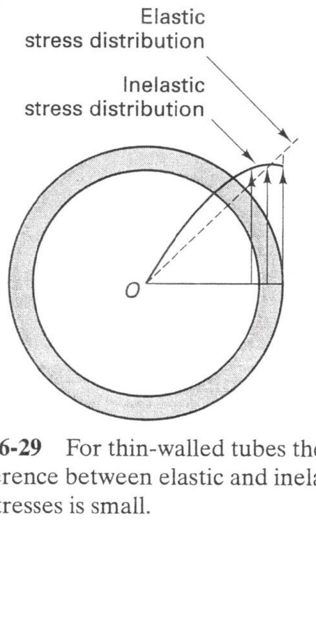

Fig. 6-29 For thin-walled tubes the difference between elastic and inelastic stresses is small.

If a shaft is strained into the inelastic range and the applied torque is then removed, every "imaginary" annulus rebounds elastically. Because of the differences in the strain paths, which cause permanent set in the material, residual stresses develop. This process will be illustrated in one of the examples that follow.

For determining the rate of twist of a circular shaft or tube, Eq. 6-13 can be used in the following form:

$$\frac{d\phi}{dx} = \frac{\gamma_{max}}{c} = \frac{\gamma_a}{\rho_a} \tag{6-28}$$

Here either the maximum shear strain at c or the strain at ρ_a determined from the stress-strain diagram must be used.

Example 6-13

A solid steel shaft of 24 mm diameter is so severely twisted that only an 8-mm-diameter elastic core remains on the inside, Fig. 6-30(a). If the material properties can be idealized, as shown in Fig. 6-30(b), what residual stresses and residual rotation will remain upon release of the applied torque? Let G = 80 GPa.

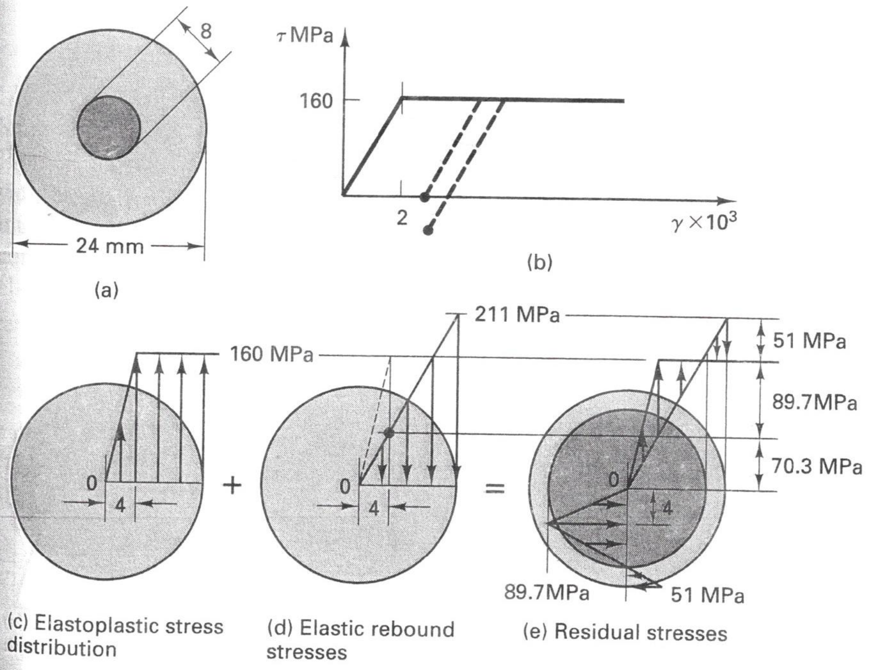

Fig. 6-30

SOLUTION

To begin, the magnitude of the initially applied torque and the corresponding angle of twist must be determined. The stress distribution corresponding to the given condition is shown in Fig. 6-30(c). The stresses vary linearly from 0 to 160 MPa when $0 \le \rho \le 4$ mm; the stress is a constant 160 MPa for $\rho > 4$ mm. Equation 6-27 can be used to determine the applied torque T. The release of torque T causes elastic stresses, and Eq. 6-3 applies; see Fig. 6-30(d). The difference between the two stress distributions, corresponding to no external torque, gives the residual stresses.

$$T = \int_A \tau\rho \, dA = \int_0^c 2\pi\tau\rho^2 \, d\rho = \int_0^4 \left(\frac{\rho}{4} 160\right) 2\pi\rho^2 \, d\rho + \int_4^{12} (160) 2\pi\rho^2 \, d\rho$$

$$= (16 + 558) \times 10^3 \text{ N} \cdot \text{mm} = 574 \times 10^3 \text{ N} \cdot \text{mm}$$

Note the small contribution to the total of the first integral.

$$\tau_{max} = \frac{Tc}{I_p} = \frac{574 \times 10^3 \times 12}{(\pi/32) \times 24^4} = 211 \text{ MPa}$$

At $\rho = 12$ mm, $\tau_{residual} = 211 - 160 = 51$ MPa.

Two alternative residual stress diagrams are shown in Fig. 6-30(e). For clarity, the initial results are replotted from the vertical line. In the entire shaded portion of the diagram, the residual torque is clockwise; an exactly equal residual torque acts in the opposite direction in the inner portion of the shaft.

The initial rotation is best determined by calculating the twist of the elastic core. At $\rho = 4$ mm, $\gamma = 2 \times 10^{-3}$. The elastic rebound of the shaft is given by Eq. 6-16. The difference between the inelastic and the elastic twists gives the residual rotation per unit length of shaft. If the initial torque is reapplied in the same direction, the shaft responds elastically.

Inelastic:

$$\frac{d\phi}{dx} = \frac{\gamma_a}{\rho_a} = \frac{2 \times 10^{-3}}{4 \times 10^{-3}} = 0.50 \text{ rad/m}$$

Elastic:

$$\frac{d\phi}{dx} = \frac{T}{I_p G} = \frac{574 \times 10^3 \times 10^3}{(\pi/32) \times 24^4 \times 80 \times 10^3} = 0.22 \text{ rad/m}$$

Residual:

$$\frac{d\phi}{dx} = 0.50 - 0.22 = 0.28 \text{ rad/m}$$

Example 6-14

Determine the ultimate torque carried by a solid circular shaft of mild steel when shear stresses above the proportional limit are reached essentially everywhere. For mild steel, the shear stress-strain diagram can be idealized to that shown in Fig. 6-31(a). The shear yield-point stress, τ_{yp}, is to be taken as being the same as the proportional limit in shear, τ_{pl}.

SOLUTION

If a very large torque is imposed on a member, large strains take place everywhere, except near the center. Corresponding to the large strains for the idealized material considered, the yield-point shear stress will be reached everywhere except near the center. However, the resistance to the applied torque offered by the material located near the center of the shaft is negligible as the corresponding ρ's are small, Fig. 6-31(b). (See the contribution to torque T by the elastic action in Example 6-13.) Hence, it can be assumed with a sufficient degree of accuracy that a constant shear stress τ_{yp} is acting everywhere on the section considered. The torque corresponding to this condition may be considered the *ultimate limit* torque. [Figure 6-31(c) gives a firmer basis for this statement.] Thus,

$$T_{ult} = \int_A (\tau_{yp}\, dA)\rho = \int_0^c 2\pi\rho^2 \tau_{yp}\, d\rho = \frac{2\pi c^3}{3}\tau_{yp} \qquad (6\text{-}29)$$

$$= \frac{4}{3}\frac{\tau_{yp}}{c}\frac{\pi c^4}{2} = \frac{4}{3}\frac{\tau_{yp} I_p}{c}$$

Since the maximum elastic torque capacity of a solid shaft is $T_{yp} = \tau_{yp} I_p/c$, Eq. 6-3, and T_{ult} is $\frac{4}{3}$ times this value, the remaining torque capacity after yield is $\frac{1}{3}$ of that at yield. A plot of torque T versus θ, the angle of twist per unit distance, as full plasticity develops is shown in Figure 6-31(c). Point A corresponds to the results found in the preceding example, line AB is the elastic rebound, and point B is the residual θ for the same problem.

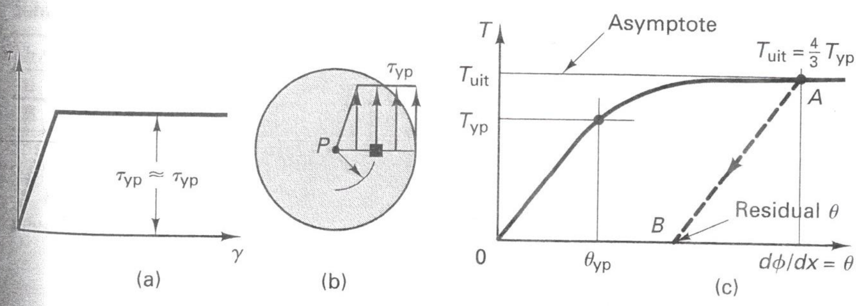

Fig. 6-31

It should be noted that in machine members, because of the fatigue properties of materials, the ultimate static capacity of the shafts as evaluated here is often of minor importance.

Part C TORSION OF SOLID NONCIRCULAR MEMBERS

6-14. Solid Bars of Any Cross Section

The analytical treatment of solid noncircular members in torsion is beyond the scope of this book. Mathematically, the problem is complex.[13] The first two assumptions stated in Section 6-3 do not apply for noncircular members. Sections perpendicular to the axis of a member warp when a torque is applied. The nature of the distortions that take place in a rectangular section can be surmised from Fig. 6-32.[14] For a rectangular member, the corner elements do not distort at all. Therefore, shear stresses at the corners are zero; they are maximum at the midpoints of the long sides. Figure 6-33 shows the shear stress distribution along three radial lines emanating from the center. Note particularly the difference in this stress distribution compared with that of a circular section. For the latter, the stress is a maximum at the most remote point, but for the former, the stress is zero at the most remote point. This situation can be clarified by considering a corner element, as shown in Fig. 6-34. If a shear stress τ existed at the corner, it could be resolved into two components parallel to the edges of the bar. However, as shears always occur in pairs acting on mutually perpendicular planes, these components would have to be met by shears lying in the planes of the outside surfaces. The latter situation is impossible as outside surfaces are free of all stresses. Hence, τ must be zero. Similar considerations can be applied to other points on the boundary. All shear stresses in the plane of a cut near the boundaries act parallel to them.

Analytical solutions for torsion of rectangular, elastic members have been obtained.[15] The methods used are beyond the scope of this book. The

[13]This problem remained unsolved until the famous French elastician B. de Saint Venant developed a solution for such problems in 1853. The general torsion problem is sometimes referred to as the St. Venant problem.

[14]An experiment with a rubber eraser on which a rectangular grating is ruled demonstrates this type of distortion.

[15]S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed. (New York: McGraw-Hill, 1970), 312. The table of coefficients that follows is adapted from this source.