Φυλλάδιο 1, άσκ. 4

$$f(z) = \bar{z}\, e^{-|z|^2}, \quad z \in \mathbb{C}$$

$$f(x+iy) = (x - iy)\, e^{-(x^2+y^2)} = \underbrace{x\, e^{-(x^2+y^2)}}_{u} + i\Big[\underbrace{-y\, e^{-(x^2+y^2)}}_{v}\Big]$$

Οι u, v είναι διαφορ. στο $\mathbb{R}^2$.

$$f(z) = \bar{z}\, e^{-z\bar{z}} \Rightarrow \frac{\partial f}{\partial \bar{z}} = e^{-z\bar{z}} + \bar{z} \cdot e^{-z\bar{z}}(-z)$$

$$= e^{-|z|^2}\left(1 - |z|^2\right)$$

$\dfrac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow |z| = 1$ δηλ. f διαφορίσιμη μόνο στα $z_0 \in \mathbb{C}$ με $|z_0| = 1$

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \bar{z}_0 \, e^{-z_0 \bar{z}_0} (-\bar{z}_0) =$$

$$= - \bar{z}_0^{\,2} \, e^{-|z_0|^2} = -\bar{z}_0^{\,2} \cdot e^{-1},$$

$\forall z_0 \in \mathbb{C}$ με $|z_0| = 1$.

$\vdash\!\dashv$

<u>Ασκ. 2</u> $f(z) = \mathrm{Log}\left(\frac{1+z}{1-z}\right)$ f διαφορ. ;

Η $w \longmapsto \mathrm{Log}\, w$ είναι διαφορ. στα $w \in \mathbb{C} \setminus (-\infty, 0]$

δηλ. <u>δεν</u> <u>είναι διαφορ.</u> στα $w \in \mathbb{C} \,|\, \mathrm{Im}\, w = 0,\ w \leq 0$

$$\frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-\bar{z}+z-|z|^2}{|1-z|^2} = \frac{1-|z|^2}{|1-z|^2} + 2i \frac{\mathrm{Im}\, z}{|1-z|^2}$$

Η f δεν είναι διαφορ. στα z με $\begin{cases} \mathrm{Im}\, z = 0 \\ |z| \leq 1 \end{cases}$

$\Rightarrow$ f διαφορ. στο $\mathbb{C} \setminus [-1,1]$.

Ασκ. 11 (i) (ε): $\alpha x + \beta y = \gamma$, $(\alpha, \beta) \neq (0,0)$

$\forall z \in A$, $f(z) \in \varepsilon$. Αν $f = u + iv$, τότε

$\forall (x,y) \in A$, $\alpha u(x,y) + \beta v(x,y) = \gamma$

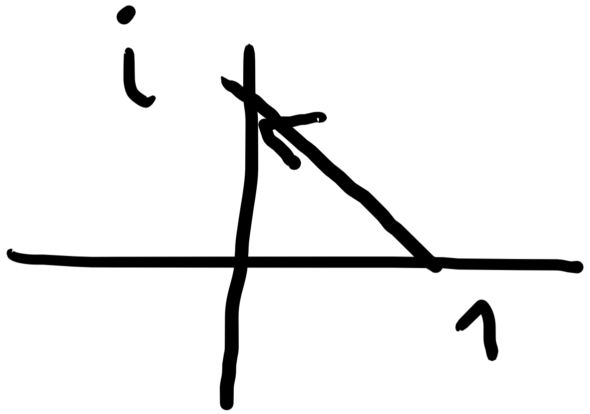
$\Rightarrow$ ~~πα~~ παραγωγ. ως $C - R$ κ.λ.π.

(ii) κύκλος C: $|z - z_0| = R$ $(R > 0)$

$\forall z \in A$, $|f(z) - z_0| = R$ $\Rightarrow$ η $g(z) = f(z) + z_0$

έχει σταθερό μέτρο στο πεδίο A κ' είναι ολόμορφη $\Rightarrow g = \text{σταθ.} \Rightarrow f = \text{σταθ.}$

ΦΥΛΛΑΔΙΟ 2



(1) <u>(iv)</u> $f(z) = \dfrac{\mathrm{Log}\, z}{z}$, $\gamma = [1, i]$

$\gamma^* \subset U = \mathbb{C} \setminus (-\infty, 0]$ κ' $F(z) = \frac{1}{2} \mathrm{Log}^2 z$ είναι ολόμορφη στο U με $F' = f$

$$\Rightarrow \int_\gamma f = F(i) - F(1) = \frac{1}{2}\left(\mathrm{Log}^2 i - \mathrm{Log}^2 1\right) = \dots$$

(v) $f(z) = \overline{z^2 e^z}$, $\quad \gamma(t) = e^{it}$, $\quad t \in [0, \pi]$,

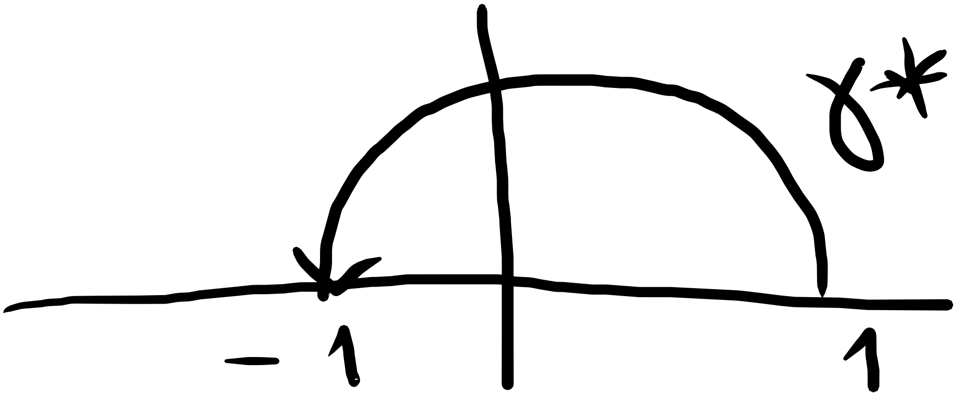
$\forall z \in \gamma^*$, $\quad |z| = 1 \Rightarrow \bar{z} = \frac{1}{z}$

$$\Rightarrow f(z) = \bar{z}^2 \cdot e^{\bar{z}} = \frac{1}{z^2} e^{1/z} = -\left(e^{1/z}\right)'$$

$$\Rightarrow \int_\gamma f = -e^{1/z}\Big|_{z=-1}^{z=1} = -\left(e - e^{-1}\right) = e^{-1} - e$$

Άσκ. 2 (i) $\quad f(z) = \frac{e^{iz}}{z^2} \quad \gamma(t) = e^{it}$, $\quad t \in [0, \pi]$

$$\left|\int_\gamma f\right| \leq \dots$$



$\forall z \in \gamma^*$, on a $z = x + iy$,

car $e^{iz} = e^{-y} \cdot e^{ix}$

$\Rightarrow |e^{iz}| = e^{-y} \leq 1$

$\Rightarrow \left| \dfrac{e^{iz}}{z^2} \right| \leq 1 \Rightarrow \left| \int_\gamma f \right| \leq 1 \cdot \underbrace{\|\gamma\|}_{= \pi}.$

Φυλλ. 4

7(i) $f(z) = \dfrac{1}{z(z-1)}$ Laurent γύρω από το $z_0 = -2$,

στον $2 < |z+2| < 3$.

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

· $\dfrac{1}{z-1} = ?$ $\dfrac{z+2}{3} = w \Rightarrow \begin{cases} |w| < 1 \\ z = 3w - 2 \end{cases}$

$\Rightarrow z - 1 = 3(w-1)$

$$\Rightarrow \frac{1}{z-1} = \frac{1}{3(w-1)} =$$

$$= -\frac{1}{3}\frac{1}{1-w} \overset{(|w|<1)}{=} -\frac{1}{3}\sum_{n=0}^{\infty} w^n$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^n}$$

$\cdot \; \frac{1}{z} = ?$ $2 < |z+2| \Rightarrow \begin{cases} w = \frac{2}{z+2}, \quad |w|<1 \\ \frac{1}{w} = \frac{z}{2} + 1 \end{cases}$

$=$

$$\Rightarrow \frac{z}{2} = \frac{1}{w} - 1 = \frac{1-w}{w}$$

$$\Rightarrow \quad z = \frac{2(1-w)}{w} \Rightarrow \frac{1}{z} = \frac{w}{2} \cdot \frac{1}{1-w} \underset{|w|<1}{=}$$

$$= \frac{w}{2} \sum_{n=0}^{\infty} w^n = \frac{1}{2} \sum_{n=0}^{\infty} w^{n+1} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2}{z+2}\right)^{n+1}$$

$$\Rightarrow \frac{1}{z} = \sum_{n=0}^{\infty} \frac{2^n}{(z+2)^{n+1}}$$

Άρα, $\forall z$ με $\underline{2 < |z+2| < 3}$,

$$f(z) = -\sum_{n=0}^{\infty} \frac{(z+2)^n}{3^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{(z+2)^{n+1}}$$

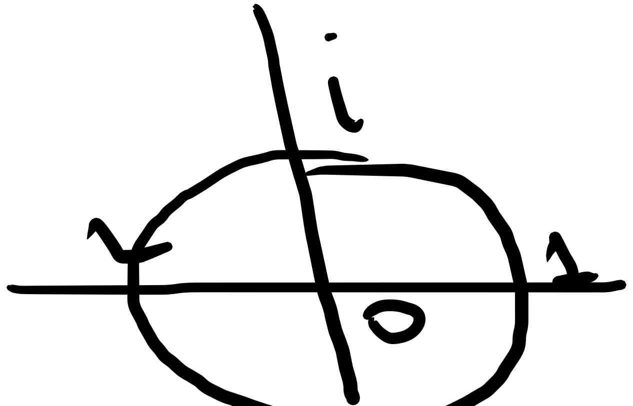
Ασκ. 12 $\int_\gamma h = ?$, $h(z) = \frac{1}{1-\cos z} + \bar{z} \cdot z^{12} \cdot \cos(1/z^3)$

$\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

$$f(z) = \frac{1}{1-\cos z}, \quad g(z) = \bar{z}\, z^{12} \cos(1/z^3)$$

$$\int_\gamma h = \int_\gamma f + \int_\gamma g$$

• $\int_\gamma f = ?$



Ανώμαλα σημεία της f: $2k\pi$, $k \in \mathbb{Z}$

Μόνο $0 \in \operatorname{int} \gamma^*$

$$\Rightarrow \int_\gamma f = 2\pi i \,\mathrm{Res}(f,0)$$

$$f(z) = \frac{1}{1-\cos z} = \frac{1}{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\right)}$$

$$= \frac{1}{\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\dots} = \frac{\varphi(z)}{z^2}$$

όπου $\varphi = 1/\psi$, $\psi(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots$

$\psi(0) = 1/2$, $\psi'(0) = 0$ $\quad \psi(0) \neq 0 \Rightarrow$ φ ολόμορφη σε περιοχή του 0 κ' $\varphi(0) = \frac{1}{\psi(0)} = 2 \neq 0$ $\Longrightarrow$ 0 <u>πόλος της f</u>

<u>τάξης 2</u>

$$\Rightarrow \operatorname{Res}(f, 0) = \frac{1}{(2-1)!} \lim_{z \to 0} \left[(z-0)^2 \cdot f(z) \right]'$$

$$= \lim_{z \to 0} \varphi'(z) = \varphi'(0) = -\frac{\psi'(0)}{\psi(0)^2} = 0$$

• $\underline{\int_\gamma g = ?}$ $\quad \forall z \in \gamma^*$, $\bar{z} = 1/z \Rightarrow g(z) = \frac{1}{z} z^{12} \cos(1/z^3)$

$$= z^{11} \cos(1/z^3).$$

οπότε, $\int_\gamma g = \int_\gamma z^{11} \cos\left(\frac{1}{z^3}\right) dz$

$$= 2\pi i \operatorname{Res}\left[z^{11} \cos\left(\frac{1}{z^3}\right), 0\right]$$

$\forall z \neq 0$, $\quad z^{11} \cos\left(\frac{1}{z^3}\right) = z^{11} \cdot \left[1 - \frac{1}{2!}\frac{1}{z^6} + \frac{1}{4!}\frac{1}{z^{12}} - \cdots\right]$

$$= z^{11} - \frac{1}{2!} z^5 + \frac{1}{4!}\frac{1}{z} - \cdots$$

$$\Rightarrow \operatorname{Res}\left(z^{11}\cos\left(\frac{1}{z^3}\right), 0\right) = \frac{1}{4!}$$

$$\Rightarrow \int_\gamma g = \frac{2\pi i}{4!} = \frac{2\pi i}{24} = \pi i/12$$

$$\Rightarrow \int_\gamma h = \int_\gamma f + \int_\gamma g = 0 + \pi i/12 = \pi i/12$$