Απαντήσεις Φυλλαδίου "ΑΟΡΙΣΤΑ ΟΛΟΚΛΗΡΩΜΑΤΑ"

1.

$$\int \frac{1+\sin x}{1-\cos x}dx = \ln\left|\sin\left(\frac{x}{2}\right)\right| - \cot\left(\frac{x}{2}\right)+c, \quad \int x\sqrt{1-x}dx = -\frac{2}{15}(3x+2)(1-x)^{3/2}+c,$$

$$\int e^x(\ln|\cos x| - \tan x)dx = e^x \ln|\cos x| + c, \quad \int \frac{dx}{\sin x} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + c, \quad .$$

$$\int \frac{dx}{\cos x} = -\ln\left|\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + c.$$

2.

$$\int x\cos x dx = \cos x + x\sin x + c, \quad \int x\sin^2 x dx = \frac{x^2 - x\sin(2x)}{4} - \frac{\cos(2x)}{8} + c,$$

$$\int x\cos x\cos(2x) = \frac{\cos(3x)}{18} + \frac{x\sin(3x)}{6} + \frac{\cos x + x\sin x}{2} + c,$$

$$\int x^2 \sin x\cos x dx = \frac{x\sin(2x)}{4} - \cos(2x)\left(\frac{x^2}{4} - \frac{1}{8}\right) + c,$$

$$\int \frac{\ln x}{x\sqrt{x}}dx = -\frac{2}{\sqrt{x}}(1+\ln x) + c, \quad \int \cos(\ln x)dx = \frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + c,$$

$$\int \text{Arcsin}x dx = x\text{Arcsin}x + \sqrt{1-x^2} + c, \quad \int \frac{x}{\sin^2 x}dx = \ln(|\sin x|) - x\cot x + c,$$

$$\int \frac{x^4}{(x^2+1)^2}dx = -\frac{x^3}{2(1+x^2)} + \frac{3x}{2} - \frac{3\text{Arctan}x}{2} + c,$$

$$\int x\text{Arctan}x dx = \frac{(1+x^2)\text{Arctan}x - x}{2} + c, \quad \int \frac{x\text{Arcsin}x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}\text{Arcsin}x + x + c.$$

3. (ii)

$$\int \sin^6 x dx = \frac{5x}{16} - \frac{5\sin x\cos x}{6} - \frac{5\sin^3 x\cos x}{24} - \frac{\sin^5 x\cos x}{6} + c,$$

$$\int \frac{1}{\cos^6 x}dx = \frac{3\sin x + 4\cos^2 x\sin x + 8\cos^4 x\sin x}{15\cos^5 x} + c.$$

4.

$$\int (1-x)^{100}x^2 dx = -\frac{(1-x)^{k+3}}{k+3} + \frac{2(1-x)^{k+2}}{k+2} - \frac{(1-x)^{k+1}}{k+1} + c, \quad k = 100,$$

$$\int \frac{x^9}{\sqrt{1-x^5}}dx = -\frac{2}{15}\sqrt{1-x^5}\,(x^5+2) + c, \quad \int \frac{dx}{x(x^7-1)} = \frac{1}{7}\ln|x^7-1| - \ln|x| + c,$$

$$\int x^5 e^{x^3} dx = \frac{x^3-1}{3}e^{x^3} + c,$$

$$\int \frac{e^{2x}}{(1+e^x)^2}dx = \ln(1+e^x) - \frac{e^x}{1+e^x} + c, \quad \int \sin^5 x dx = -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + c,$$

$$\int \frac{\sin(2x)\cos^2 x}{(1+\cos^2 x)^2}dx = -\ln(1+\cos^2 x) - \frac{1}{1+\cos^2 x}+c, \quad \int e^{2x}\cos(e^x)dx = \cos(e^x) + e^x\sin(e^x) + c,$$

$$\int \frac{\ln(\ln x)}{x}dx = \ln x(\ln(\ln x) - 1)+c, \quad \int \sin(\sqrt{x})dx = 2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x})+c,$$

$$\int \frac{\mathrm{Arcsin}(\sqrt{x})}{\sqrt{1-x}}dx = -2\sqrt{1-x}\ \mathrm{Arcsin}(\sqrt{x}) \ + \ 2\sqrt{x} + c.$$

5.

$$I = \frac{x \ - \ \ln|\cos x + \sin x|}{2} + c, \quad J = \frac{x \ + \ \ln|\cos x + \sin x|}{2} + c.$$

6. (ii)

$$\int \tan^4 x dx = \frac{\tan^3 x}{3} \ - \ \tan x \ + \ x \ + \ c,$$

$$\int x\tan^4 x dx = -\frac{4}{3}\ln|\cos x| \ + \ \frac{x\tan^3 x}{3} \ - \ \frac{\tan^2 x}{6} \ - \ x\tan x \ + \ \frac{x^2}{2} \ + \ c,$$

$$\int \frac{x^4 \mathrm{Arctan} x}{x^2+1}dx = \frac{2\ln(1+x^2)}{3} \ + \ \frac{x^3 u(x)}{3} \ - \ \frac{x^2}{6} \ - \ xu(x) \ + \ \frac{u(x)^2}{2} + c,$$

όπου $u(x) = \mathrm{Arctan}\ x$.

7.

$$\int \frac{x^3+x^2+x-1}{(x+1)^2(x^2+1)}dx = \frac{\ln(1+x^2)}{2} \; + \; \frac{1}{x+1} + c,$$

$$\int \frac{x^5+2}{x^2-1}dx = \frac{3\ln|x-1| \; - \; \ln|x+1| \; + \; x^2}{2} \; + \; \frac{x^4}{4} \; + c,$$

$$\int \frac{dx}{x^4-4} = \frac{\sqrt{2}}{16}(\ \ln|x-1| \; - \; \ln|x+1| \; - \; 2\text{Arctan}x\ ) + c.$$

$$\int \frac{-x^3+2x^2-3x+1}{x(x^2+1)^2}dx = \ln x - \frac{1}{2}\ln(1+x^2) - 2\text{Arctan}x \; - \; \frac{2x+1}{2(1+x^2)} + c,$$

$$\int \frac{x^3+8}{x^3-2x^2-x+2}dx = x \; - \; \frac{9}{2}\ln|x-1| \; + \; \frac{7}{6}\ln|x+1| \; + \; \frac{16}{3}\ln|x-2| + c,$$

$$\int \frac{x+2}{(x+1)^3(x-2)}dx = \frac{4}{27}\ln\left|\frac{x-2}{x+1}\right| \; + \; \frac{8x+11}{18(x+1)^2} + c,$$

$$\int \frac{3x-5}{(x^2-2x+5)^2}dx = -\frac{1}{8}\text{Arctan}\left(\frac{x-1}{2}\right) \; - \; \frac{x+5}{4(x^2-2x+5)} + c,$$

$$\int \frac{dx}{x^4+4} = \frac{1}{8}\left[\ \ln\frac{x^2+2x+2}{x^2-2x+2} + \text{Arctan}(x+1) + \text{Arctan}(x-1)\ \right] + c.$$

(Υπόδειξη: Η ανάλυση σε απλά κλάσματα εδώ μπορεί να πάρει τη μορφή

$$\frac{1}{x^4+4} = \frac{Ax+B}{x^2+2x+2} \; + \; \frac{-Ax+B}{x^2-2x+2}, \quad A, B \in \mathbb{R}.)$$

8.

$$\int \frac{\sin^5 x}{\cos^4 x} dx = -\cos x - \frac{2}{\cos x} + \frac{1}{3\cos^3 x} + c,$$

$$\int \frac{\sin^2 x}{\cos^3 x} dx = \frac{\sin x}{2\cos^2 x} - \frac{1}{4} \ln \left| \frac{1+\sin x}{1-\sin x} \right| + c,$$

$$\int \frac{dx}{3\sin^2 x + 5\cos^2 x} = \frac{1}{\sqrt{15}} \text{Arctan} \left(\frac{\sqrt{3}\tan x}{\sqrt{5}} \right) + c,$$

$$\int \frac{\cos x}{2\cos x + \sin x + 3} dx = 4\ln(1+y^2) - 6y + 16\text{Arctan} y - 2y^2 - 2y^3/3 + c, \quad y = \tan(x/2),$$

$$\int \frac{\cos^4 x}{\sin^3 x} dx = -\frac{\cos^3 x}{2\sin^3 x} - \frac{3\cos x}{2} + \frac{3}{4} \ln \left| \frac{1+\cos x}{1-\cos x} \right| + c,$$

$$\int \sin^5 x \cos^2 x dx = \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3} - \frac{\cos^7 x}{7} + c,$$

$$\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{\tan^3 x}{3} + 2\tan x - \frac{1}{\tan x} + c,$$

$$\int \frac{\sin x}{1+\sin x + \cos x} dx = \frac{1}{2} [\, x/2 - \ln|\sin(x/2) + \cos(x/2)|\,] + c$$

(Υπόδειξη: Αντί της μετατροπής σε ολοκλήρωμα ρητής συνάρτησης, μπορείτε εναλλακτικά να θέσετε $x = 2t$, οπότε προκύπτει το ολοκλήρωμα I της άσκησης 5),

$$\int \frac{e^x}{e^{2x}-4}dx = \frac{1}{4}\ln\left|\frac{e^x-2}{e^x+2}\right| + c, \quad \int \frac{dx}{\sqrt{1+e^x}} = \ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + c,$$

$$\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \ln(e^x+e^{-x}) + c,$$

$$\int \sqrt{1+\frac{1}{x}}\,dx = \frac{y}{y^2-1} \ - \ \frac{1}{2}\ln\left(\frac{y-1}{y+1}\right) + c, \qquad y = \sqrt{1+\frac{1}{x}}\ .$$

9.

$$I+J = \frac{x}{8} \ - \ \frac{\sin(4x)}{32} + c, \quad I-J = \frac{\sin^3(2x)}{24} + c,$$

$$I = \frac{x}{16} \ - \ \frac{\sin(4x)}{64} \ + \ \frac{\sin^3(2x)}{48} + c,$$

$$J = \frac{x}{16} \ - \ \frac{\sin(4x)}{64} \ - \ \frac{\sin^3(2x)}{48} + c.$$