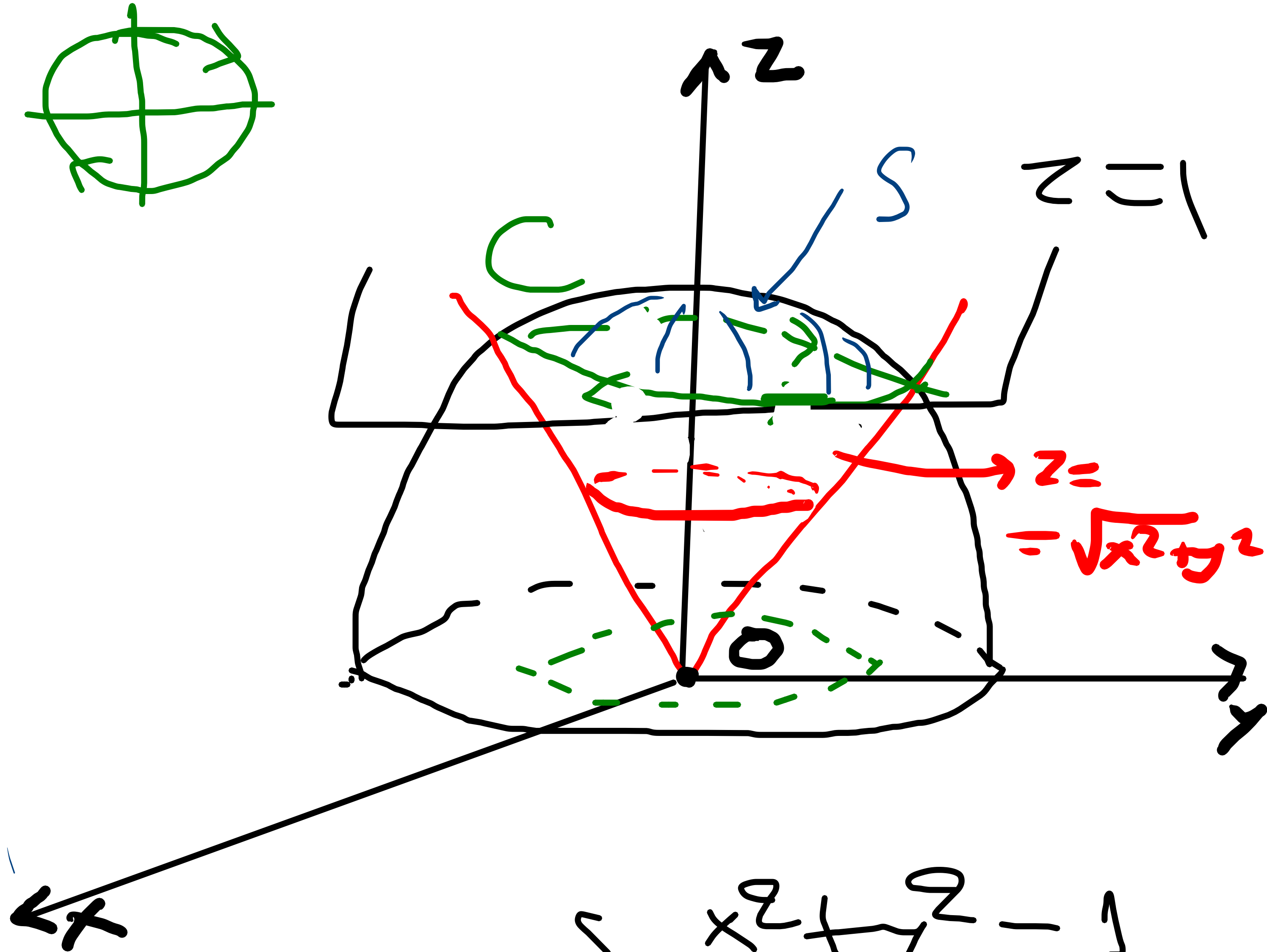
Ασκ.5, Θεωρ. "STOKES"

$\vec{F} = (-y,\ x,\ z^2 + xy)$,

$S: \begin{cases} x^2 + y^2 + z^2 = 2 \\ z \geq \sqrt{x^2+y^2} \end{cases}$



$C: \begin{cases} x^2 + y^2 + z^2 = 2 \\ z^2 = x^2 + y^2 \end{cases}$

$\Longrightarrow \quad z^2 = 1 \quad \Rightarrow \quad \boxed{z = 1}$

$C: \begin{cases} x^2 + y^2 = 1 \\ z = 1 \end{cases}$

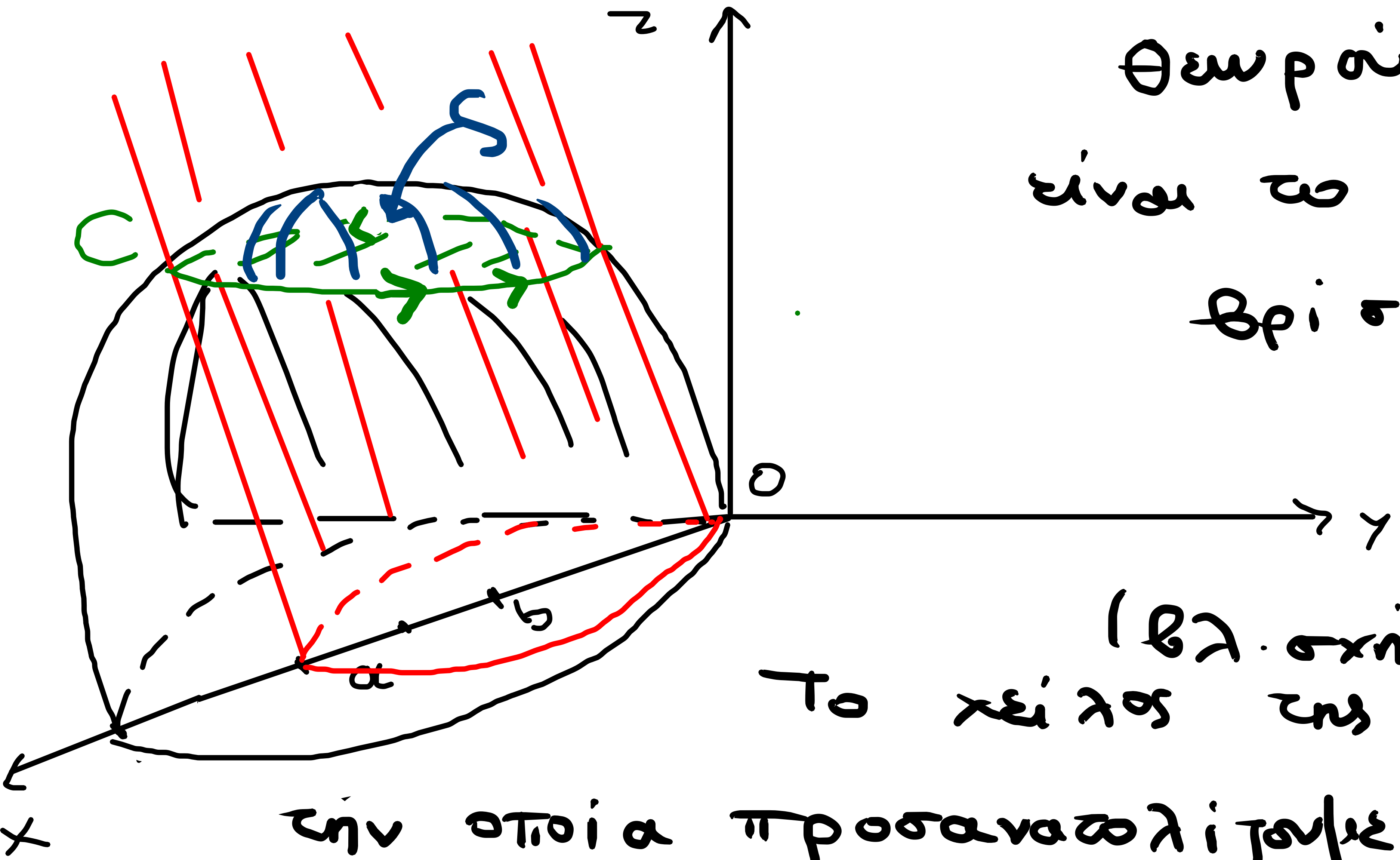
ροή του $\text{rot}\vec{F}$ μέσου της S $\underline{\underline{(\Theta.\ \text{Stokes})}}$

$= \int_C \vec{F}$ (όπου η C προσανατολ. όπως στο σχήμα)

$\left[\underline{\underline{z=1, dz=0}} \text{ πάνω στην } C \right] \int_C (-y\,dx + x\,dy) = -\int_{-C} (-y\,dx + x\,dy) =$

$$\overset{\text{(由 Green)}}{=} -\iint\limits_{x^2+y^2\le 1} 2\,dx\,dy = -2\pi.$$

Άσκ. 3, φυλλάδιο "STOKES"



Θεωρούμε την επιφάνεια S που είναι το τμήμα της σφαίρας που βρίσκεται μέσα στον κύλινδρο, δηλ.

$$S: \begin{cases} x^2+y^2+z^2 = 2ax, & z \geq 0 \\ x^2+y^2 \leq 2bx \end{cases}$$

(βλ. σχήμα — μπλε χρώμα).

Το χείλος της S είναι η C την οποία προσανατολίζουμε όπως στο σχήμα.

$S: \Delta \to \mathbb{R}^3$, $\Delta = \{(u,v): u^2+v^2 \leq 2bu\}$,

$$S(u,v) = (u, v, f(u,v)), \quad f(u,v) = \sqrt{2au - u^2 - v^2}, \ (u,v) \in \Delta.$$

(συνθ. $a > b$).

Θ. Stokes $\Rightarrow \int_C \vec{F} = \iint_\Delta \operatorname{rot}\vec{F} \cdot \vec{n} \, du\,dv.$

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2+z^2 & z^2+x^2 & x^2+y^2 \end{vmatrix} = (2y-2z, \ 2z-2x, \ 2x-2y).$$

$$S_u \times S_v = (-f_u, -f_v, 1) = \left(\frac{2u-2a}{2f}, \ \frac{2v}{2f}, \ 1\right) = \left(\frac{u-a}{f}, \ \frac{v}{f}, 1\right)$$

$$(S_u \times S_v)\cdot S(u,v) = \frac{u^2 - au + v^2 + f^2}{f} = \frac{au}{f} \geq 0$$

$\forall (u,v) \in \Delta \implies S_u \times S_v$ "δείχνει" προς τα έξω,

$$\Rightarrow \int_C \vec{F} = 2\iint_\Delta (v-f,\ f-u,\ u-v)\cdot\left(\frac{u-a}{f},\ \frac{v}{f},\ 1\right) du\,dv$$

$$= 2\iint_\Delta \frac{(v-f)(u-a) + v(f-u) + (u-v)f}{f}\, du\,dv$$

$$= 2\iint_\Delta \frac{\cancel{vu} - av - \cancel{uf} + af + \cancel{vf} - \cancel{uv} + \cancel{uf} - \cancel{vf}}{f} = 2\iint_\Delta \frac{a(f-v)}{f}\, du\,dv$$

$$= 2a\iint_\Delta du\,dv - 2a\iint_\Delta \frac{v}{f}\, du\,dv = 2\pi a b^2 - 2a\iint_\Delta \frac{v}{f}\, du\,dv$$

Αλλά, $\iint_\Delta \frac{v}{f}\,du\,dv \overset{u=r\cos\varphi}{\underset{v=r\sin\varphi}{=}} \int_0^{2\pi}\int_0^b r^2 g_r(\varphi)\,dr\,d\varphi$

όπου $g_r(\varphi) = \sin\varphi \Big/ \sqrt{2(a-b)(b+r\cos\varphi)+b^2-r^2}$.

Παρατηρούμε ότι $g_r(2\pi-\varphi) = -g_r(\varphi)$

$$\Rightarrow \int_\pi^{2\pi} g_r(\varphi)\,d\varphi \overset{\theta=2\pi-\varphi}{=} \int_\pi^0 g_r(2\pi-\theta)(-d\theta) = -\int_0^\pi g_r(\theta)\,d\theta$$

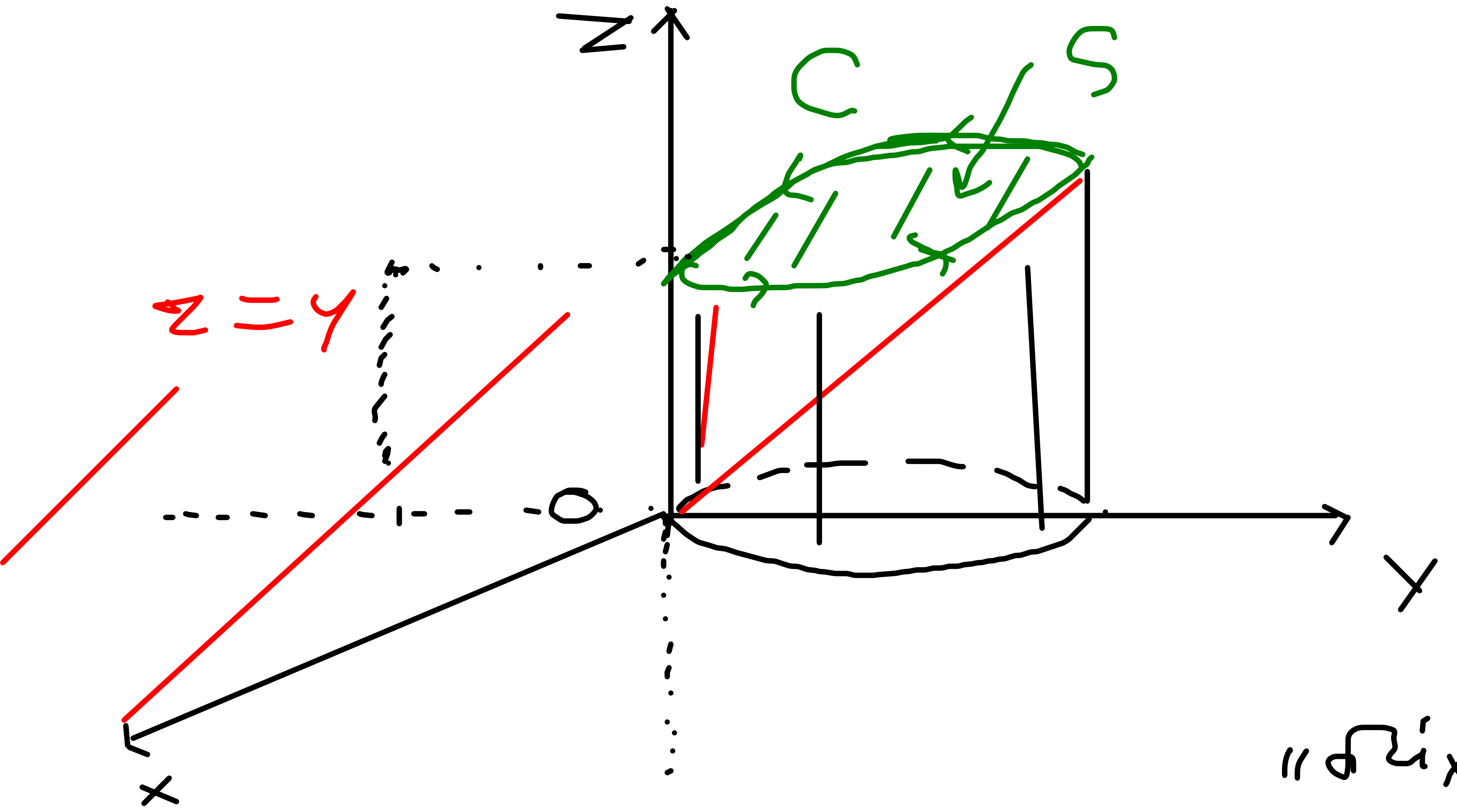
$$\Rightarrow \int_0^{2\pi} g_r(\varphi)\,d\varphi = 0 \Rightarrow \iint_\Delta \frac{v}{f}\,du\,dv = \int_0^b r^2\left[\int_0^{2\pi} g_r(\varphi)\,d\varphi\right]dr$$

$$= 0.$$

Άσκ. 2, φυλλάδιο "Stokes"

$\vec{F} = (y+z,\ z+x,\ x+y)$,

$C : \begin{cases} x^2+z^2 = 2y, & z \geq 0 \\ y = z \end{cases}$



$S : \Delta \to \mathbb{R}^3$

$S(u,v) = (u, v, v)$

$\Delta = \{(u,v) \mid u^2+v^2 \leq 2v\}$

$S_u \times S_v = (0, -1, 1)$

$S(0,0) = (0,0,0)$

"δείχνει" προς τα έξω

$$\vec{n}(u,v) = (0, -1, 1)$$

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ y+z & z+x & x+y \end{vmatrix} = (0, 0, 0)$$

$$\int_C \vec{F} = \iint_\Delta \operatorname{rot}\vec{F} \cdot \vec{n} \; du\, dv = 0.$$