Another way to describe the incompressibility and inextensibility conditions, by using the strain energy density $\underline{W}$

$$\frac{\partial \underline{W}}{\partial \underset{\sim}{C}} : \underset{\sim}{C}^{-1} = 0 \quad , \quad \frac{\partial \underline{W}}{\partial \underset{\sim}{C}} : \underset{\sim}{M} = 0$$

Zero contribution to the energy production

Incompressibility — Inextensibility

Equivalent to

$$I_3 = 1 \qquad I_4 = 1$$

Another expression for the I_4 "invariant"

$$I_4 = \underset{\sim}{C} : \underset{\sim}{M} = |\underset{\sim}{F} \cdot \vec{m}|^2 \neq 1 \text{ for deformable fibers}$$

$I_4 - 1 > 0$: extension

We can also write

$$I_4 = |\underset{\sim}{U} \cdot \underline{m}|^2$$

$\underset{\sim}{U}$: pure stretch right tensor where $\underset{\sim}{C} = \underset{\sim}{U}^2$

$\underset{\sim}{U}$ emerges in the polar decomposition theorem

$$\underset{\sim}{F} = \underset{\sim}{R} \cdot \underset{\sim}{U}$$

$\underset{\sim}{R}$ → pure rotation tensor

$\underset{\sim}{U}$ → pure stretch tensor

We also have the I_5 "invariant".

$$I_5 = |\underset{\sim}{U}^2 \cdot \underline{m}|^2$$

I_4 and I_5 are not equivalent

Possible Form of $\underline{W}$ for a material with a ground substance (matrix) reinforced with fibers

$$\underline{W} = \underbrace{\underline{W}^{iso}(I_1, I_2, I_3)}_{\text{ground substance}} + \underbrace{W^{aniso}(I_4, I_5)}_{\text{fibers}}$$

Possible form of $\underline{W}^{aniso}$

$$\underline{W}^{aniso}(I_4, I_5) = \underline{W}^{aniso}_{(4)}(I_4) + W^{aniso}_{(4-5)}(I_4, I_5)$$

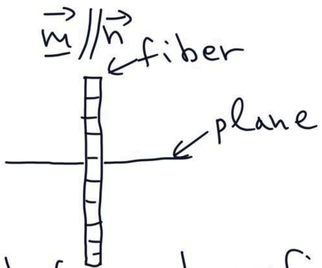
I_5 is also related to the crosslinking between individual neighbouring fibers

$$W^{aniso}_{(4)}(I_4) = W^{aniso}_{(4)}\left(\langle I_4 - 1\rangle^{\alpha}\right)$$

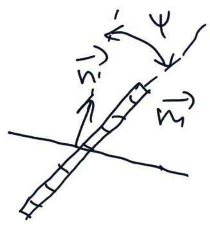
α: appropriate exponent in order to keep the stress continuous when the fibers are under compression (buckle)

The fibers may contribute ~~to~~ to the deformation of the composite material, even when they buckle (are compressed)

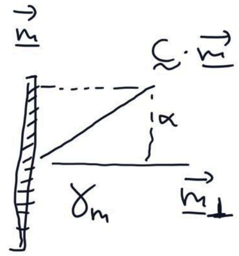
I_5 may refer to fiber-matrix interaction.



undeformed configuration
$\vec{m}$: unit directional vector along the straight fiber
$\vec{n}$: unit normal to the plane



deformed configuration
$\vec{m}'$: directional vector along the deformed fiber



Obviously $\cos\psi = \vec{m}' \cdot \vec{n}' = \sqrt{\dfrac{I_3}{I_4(I_5 - I_1 I_4 + I_2)}}$

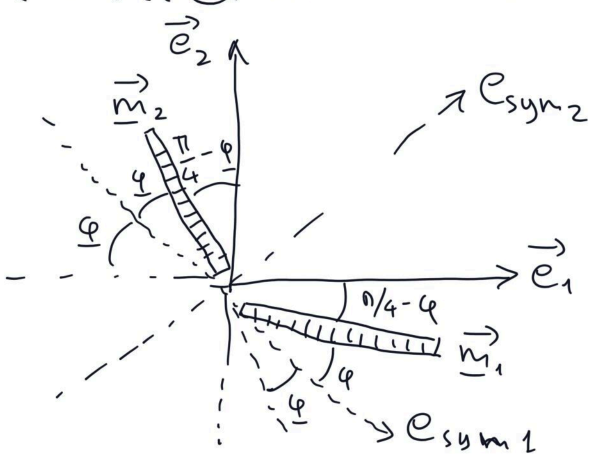
So we may write

$$W^{aniso}_{(4-5)}(I_1, I_2, I_3, I_4, I_5) = f^{(4)}(I_4)\tan^2\psi$$

For fiber-fiber interaction or cross-link intensity

$$C \cdot \vec{m} = \alpha\, \vec{m} + \gamma_m \vec{m}_\perp \quad \text{where } \gamma_m^2 = I_5 - I_4^2$$

Two families of fibers



If $\vec{m}_2, \vec{m}_1$ families are mechanically equivalent and have the same density spacially then we have an orthotropic material with respect to the $\vec{e}_{sym_1}, \vec{e}_{sym_2}$ axes. Otherwise we have a clinotropic material with two (unrelated between them) preferable directions.

Now we have the 2^{nd} order directional tensors

$$\underset{\sim}{M}_1 = \vec{m}_1 \otimes \vec{m}_1 \quad , \quad \underset{\sim}{M}_2 = \vec{m}_2 \otimes \vec{m}_2$$

$$\underset{\sim}{M}_s = \underset{\sim}{M}_1 \cdot \underset{\sim}{M}_2 + \underset{\sim}{M}_2 \cdot \underset{\sim}{M}_1 \qquad \text{s: structure}$$

New set of invariants related to the 2 families of fibers

$$I_1 = tr\,\underset{\sim}{C} \qquad I_1 = C_{11} + C_{22} + C_{33}$$

$$I_2 = \frac{1}{2}\left(tr^2\underset{\sim}{C} - tr\,\underset{\sim}{C}^2\right) \qquad I_2 = \ldots\ldots$$

$$I_4 = \underset{\sim}{C} : \underset{\sim}{M}_1$$

$$I_5 = \underset{\sim}{C}^2 : \underset{\sim}{M}_1$$

$$I_6 = \underset{\sim}{C} : \underset{\sim}{M}_2$$

$$I_7 = \underset{\sim}{C}^2 : \underset{\sim}{M}_2$$

$$I_s = \underset{\sim}{C} : \underset{\sim}{M}_s$$

For an incompressible material with two inextensible families of fibers, we can write

$$\det \underset{\sim}{C} - 1 = 0 \qquad \text{for incompressibility}$$

$$I_4 - 1 = \vec{m}_1 \cdot \underset{\sim}{C} \cdot \vec{m}_1 - 1 = 0 \qquad \text{for inextensibility of the } 1^{st} \text{ family of fibers}$$

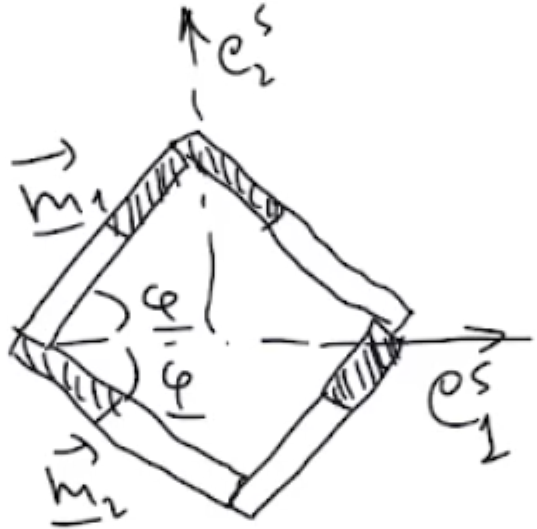
$$I_6 - 1 = \vec{m}_2 \cdot \underset{\sim}{C} \cdot \vec{m}_2 - 1 = 0 \qquad \text{for inextensibility of the } 2^{nd} \text{ family of fibers}$$

I_s: is related to energy due to interlamellar normal deformation of shear

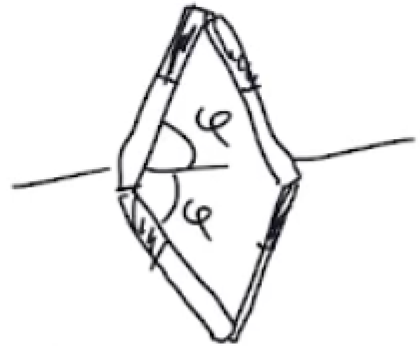
Woven fibers

The sense (direction) of the fibers maters

See picture in p. 796



Positive shear

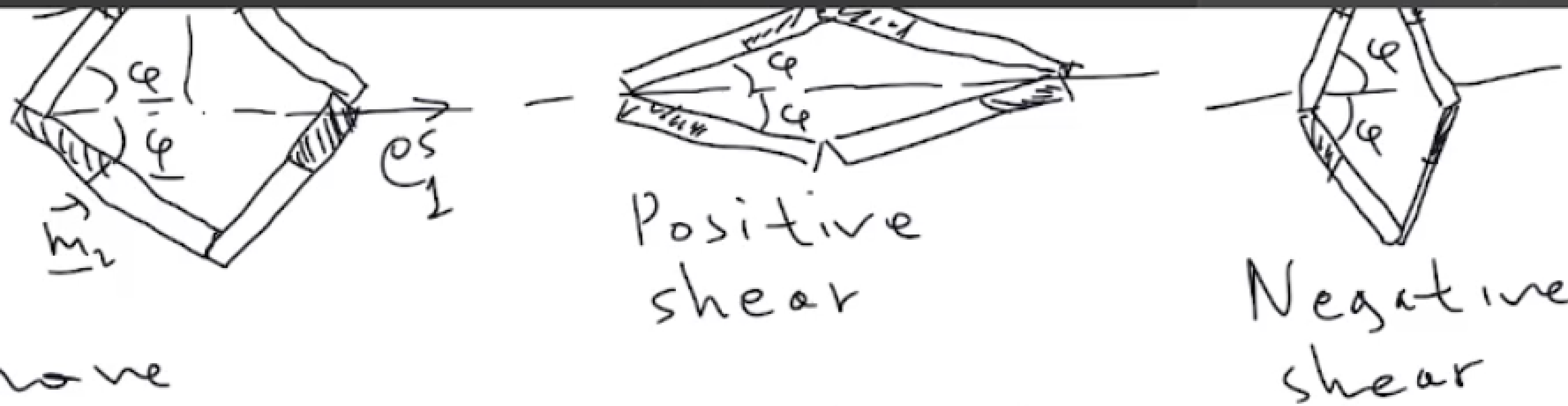


Negative shear

We have

$$\underset{\sim}{M}_{12} = \vec{m}_1 \otimes \vec{m}_2 \quad , \quad \underset{\sim}{M}_{21} = \vec{m}_2 \otimes \vec{m}_1$$

Define new tensor (for interaction)



We have

$$\underset{\sim}{M}_{12} = \underline{\vec{m}}_1 \otimes \underline{\vec{m}}_2 \quad , \quad \underset{\sim}{M}_{21} = \underline{\vec{m}}_2 \otimes \underline{\vec{m}}_1$$

Define new tensor (for interaction)

$$\underset{\sim}{X}\left(\underset{\sim}{M}_{12}, \underset{\sim}{M}_{21}, \underset{\sim}{C}\right)$$

Introduce new invariant

$$\mathrm{II}_s = \frac{1}{2} \underset{\sim}{X} : \underset{\sim}{C}$$

Introduce new invariant

$$\mathbb{II}_s = \frac{1}{2} \underset{\sim}{X} : \underset{\sim}{C}$$

The new invariants are added to the $I_1 - I_7 - I_s$ list

$$\mathbb{II}_s = a_{12}\, \underset{\sim}{C}^2 : \underset{\sim}{\underline{M}}_{12} + a_{21}\, \underset{\sim}{\underline{C}}^2 : \underset{\sim}{\underline{M}}_{21}$$

$$I_9 = -\, \underline{\vec{m}}_1 \cdot \underline{\vec{m}}_2 = \cos 2\underline{\varphi}$$

Now

$$I_s = 2\, \underline{\vec{m}}_1 \cdot$$

Now

$$I_5 = 2\,\vec{\underline{m}}_1 \cdot \underset{\sim}{C} \cdot \vec{\underline{m}}_2$$

$\underline{W}$ in terms of isochoric invariants

$$\underset{\sim}{C}_{iso} = I_3^{1/3}\,\underset{\sim}{C} \qquad J = \det \underset{\sim}{F}$$

$I_1^{iso} \ldots I_6^{iso}$ are defined

$$\underline{W} = \underline{W}^{vol}(J) + W^{shear}\left(I_1^{iso}, I_2^{iso}\right) + \underline{W}^{fibrils}\left(I_4^{iso}, I_6^{iso}\right)$$