Strain energy density functions, possibly applicable to the annulus fibrosus

An example $\underline{W}$ taken from tendons

$$\underline{W} = k_1 I_1^2 + K_2 I_2 + k_3 I_1 I_4 + k_4 I_4 + k_5 I_5$$

Refers to a compressible material (contains the Invariant I_3 which is associated to volume change). "Invariants" I_4, I_5 are related to a single fiber direction (single family of parallel, straight (in the undeformed configuration) fibers)

For two families of parallel fibers where the interaction (crosslink) between the fibers is taken into account, we can write

$$\underline{W} = \underline{W}^{iso} + \underline{W}^{crosslink} + \underline{W}^{fibrils}$$

iso : isotropic part

$$\underline{W}^{iso} = \underline{W}^{iso-volume}\left(\left(I_3 - I_3^{-1}\right)^2\right) + \underline{W}^{iso-deviatoric}\left(I_1 I_3^{-1/3} - 3\right)$$

$$\underline{W}^{crosslink} = \underline{W}^{crosslink}\left(I_5 + I_7 - I_4^2 - I_6^2\right)$$

$$\underline{W}^{fibrils} = \underline{W}^{fibrils}\left(I_4 + I_6\right).$$

Growth of biological tissues

Important in tissue engineering (e.g. production of cartilage in the laboratory)

Important in tumor expansion.

Related to mass generation (or depletion) which usually takes place along with deformation

Example Equation for degeneration of a scaffold

$$\frac{dm_{sc}(t)}{dt} + \frac{1}{\tau_{sc}} m_{sc}(t) = 0$$

m_{sc}: mass of scaffold

τ_{sc}: characteristic time

$$\Rightarrow m_{sc}(t) = m_{sc}(0)\, e^{\left(-\frac{t}{\tau_{sc}}\right)}$$

m_{sc}: initial condition

Similarly for the ECM (extracellular matrix) that is produced in the scaffold, we can write

$$\frac{d}{dt}\left(m_i(t) - m_i(\infty)\right) + \frac{1}{\tau_i}\left(m_i(t) - m_i(\infty)\right) = 0$$

τ_i: characteristis time for the ith component of the ECM (

$i = 1 = c$ refers to collagen

$i = 2 = pg$ refers to PG's

2 components of ECM in articular cartilage

$$\Rightarrow \frac{m_i(t) - m_i(\infty)}{m_i(0) - m_i(\infty)} = e^{-\left(\frac{t}{\tau_i}\right)}$$

For simultaneous mass change and deformation, we may write

$$\underset{\sim}{F} = \underset{\sim k}{F}^{e} \cdot \underset{\sim k}{F}^{g}$$

e: related to elastic deformation

g: related to mass growth

$$\vec{x} = \underset{\sim}{F} \cdot \vec{X}$$

The stresse are additive between components.

$$\sigma^s = \sum_{k \in (c, PG)} \sigma^k$$

Flow diagram for collagen

Initial configuration

Current (final) configuration

$$\begin{tikzcd}
\underline{\sigma}^c \arrow[rr, "\underset{\sim}{F} = F_c^e \cdot F_c^g"] \arrow[dr, "\rho^c", "F_c^g"'] & & \sigma^c \\
& \sigma_k^{c?} = 0 \arrow[ur, "F_c^e"'] &
\end{tikzcd}$$

k: intemediate stage

Similar diagram for the PG's.
The overall (total) stress in the medium is

$$\sigma^s = \sigma^c + \sigma^{PG}$$

at all stages

A more detailed diagram

final stress for component k

$$\begin{tikzcd}
\text{Initial configuration} \arrow[d, "\text{break in pieces}"'] & & T_k \ \text{current configuration} \\
\underset{\sim}{I} = 0 \arrow[r, "F^g_{\sim k}", "\text{grow the pieces}"'] & T_{kk} = 0? \arrow[ur, "F^e_k = F^\ell_k F^a_k"] \arrow[r, "F^a_k", "\text{assemble the pieces}"'] & T^R_k \ \text{residual stress} \arrow[u, "F^\ell_k \ \text{load}"']
\end{tikzcd}$$

The velocity gradient tensor is used in the description of growth

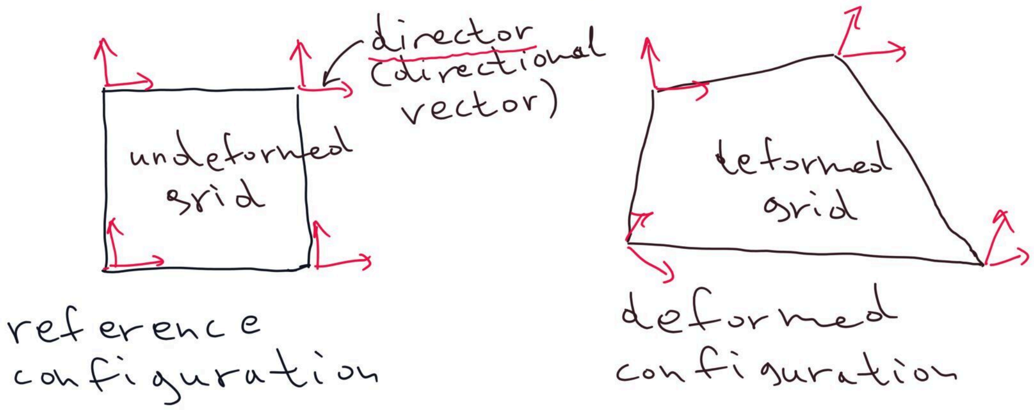
$$\underset{\sim}{L} = \frac{d\underset{\sim}{F}}{dt} \cdot \underset{\sim}{F}^{-1}$$

Growth may happen: 1) Under constant volume, with increase of mass
2) Under constant mass, with decrease of volume

Growth may happen with restructuring - remodellin. A remodelling variable

$$\underline{r}^\kappa = \frac{m^\kappa}{\underline{\rho}^\kappa} - \det \underset{\sim}{F}^g_\kappa$$

Growth may have preferred direction



Directors are not involved in the description of the deformation process. However they are necessary for the description of the growth process.

Constitutive equation for deformation and growth (w.r.t the free energy $\underline{E}$)

Stress-strain law $\quad \underline{\underline{\Sigma}} = \sum_{\kappa \in S^*} \underset{\sim}{F}^{-1} \cdot \frac{\partial E}{\partial \underset{\sim}{F}^e_\kappa} \cdot \left(\underset{\sim}{F}^g_\kappa\right)^{-T} + \underline{\underline{\Sigma}}_{visc}$

S: solid

visc: viscous losses (e.g viscoelastic material)

Entropy $\quad \underline{S} = \frac{\partial E}{\partial T}$

chemical potentials $\quad \mu_\kappa = \frac{\partial E}{\partial m^\kappa}$

dissipative mechanisms $\quad X_n = \frac{\partial E}{\partial \xi_n} - \beta^{(d)}_n$

remodelling $\quad \underline{R}_\kappa = \frac{\partial E}{\partial \underline{r}^\kappa} - \underline{\rho}^{\kappa}\tilde{e}_\kappa - \beta^{(r)}_\kappa$

ξ_n: dissipation variable

$\beta^{(d)}_n$: dissipation production variable

$\tilde{e}_\kappa$: director vector

$\beta^{(r)}_\kappa$: remodelling source variable

E.g

4th order tensor

$$\underline{\underline{\Sigma}}_{visc} = t_v \, \underline{\underline{E}}^v : \frac{d\underset{\sim}{E}_G}{dt}$$

constant $\quad$ "stiffness" tensor